Math 1

Calculus 1 and Linear Algebra

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Contents

1. Logic, sets and functions

When dealing with mathematics in the field of engineering we first need to get a common understanding on the basic concepts. Some (or even most) of the material presented here may be already familiar to you. In any case we must define a common language to express more advanced concepts later on.

1.1. Logic

Definition 1.1 (Statement)**.** A *statement* is a sentence with the following properties:

1. It states a fact.
2. It is either *true* or *false*.

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Example 1.1.

1. Christmas is in December. ✓
   (states a fact and is true)
2. Bonn is the capital of Germany. ✓
   (states a fact and is false)
3. Are you hungry? ✗
   (doesn't state a fact)
4. I shave every man in my village. ✓
   (states a fact and is true or false)
5. I shave every man in my village who doesn't shave himself. ✗
   (states a fact but is impossible to resolve)
6. This statement you are reading is a lie. ✗
   (states a fact but is impossible to resolve)

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Remark: For convenience we often write 0 for *false* and 1 for *true*.

Definition 1.2 (Compound statements)**.** Let a and b be statements, then we form the following *compound statements*:

name	in words	notation
negation	"not a"	$\overline{a}$ or $\neg a$
conjunction	"a and b"	$a \wedge b$
disjunction	"a or b"	$a \vee b$
implication	"a implies b"	$a \Rightarrow b$ or $a \rightarrow b$
equivalence	"a equals b"	$a \Leftrightarrow b$ or $a \leftrightarrow b$

and define them by the following truth table:

a	b	$\overline{a}$	$a \wedge b$	$a \vee b$	$a \Rightarrow b$	$a \Leftrightarrow b$
0	0	1	0	0	1	1
0	1	1	0	1	1	0
1	0	0	0	1	0	0
1	1	0	1	1	1	1

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Example 1.2. As an example for implication let us look at the following statement: "If it rains the street will be wet." It may be written in the following table:

weather	street	statement is
not raining	not wet	true
not raining	wet	true
raining	not wet	false
raining	wet	true

It appears unfamiliar that the first two rows support the statement and that only the third row results in *false*. However, the statement does not state that it does not rain or that the street might not get wet by some other means. Hence, the statement remains true except for the combination of rain and a dry street. ◁

Definition 1.3 (Necessity and sufficiency)**.**
A *sufficient condition* of a statement to be true assures the statement is true too. I.e. a sufficient condition implies the statement.

$$\text{sufficient condition} \quad \Rightarrow \quad \text{statement}$$

A *necessary condition* of a statement must be true for the statement to be true too. I.e. the statement implies the necessary condition.

$$\text{statement} \quad \Rightarrow \quad \text{necessary condition}$$

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Example 1.3. We go back to the previous example: "If it rains the street will be wet." The rain is a sufficient, but not necessary condition for the street to be wet. Pouring out a bucket of water has the same effect without rain ...

"You need a chequerboard to play chess." Having a chequerboard is a necessary but not sufficient condition to play chess. You also need 16 chessmen, a second chess player etc.

The date October third is a necessary and sufficient condition for the *German Unification Day*. I.e. "If the date is October third we have the German Unification day." and "On the German Unification day we have October third." ◁

Remark: If a statement has a necessary and sufficient condition, then statement and condition are equivalent:

$$\text{condition} \quad \Leftrightarrow \quad \text{statement}$$

Another term is *if and only if*, in short *iff*.

Definition 1.4 (Associativity)**.** If for a binary operator $\odot$ we have

$$(a \odot b) \odot c = a \odot (b \odot c) \qquad \text{for all } a, b, c$$

we say the operator $\odot$ is *associative*, i.e. it fulfils the law of *associativity*. ◁

Remark: We used the term *binary operator*. It is an operator that requires two operands. E.g. the operator '+' for addition requires as operands two summands.

In contrast an *unary operator* requires only one operand. E.g. the operator $\sqrt{\cdot}$ for the square root needs only the radicand as its operand.

Definition 1.5 (Commutativity)**.** If for a binary operator $\odot$ we have

$$a \odot b = b \odot a \qquad \text{for all } a, b$$

we say the operator $\odot$ is *commutative*, i.e. it fulfils the law of *commutativity*. ◁

Definition 1.6 (Distributivity)**.** Let $\odot$ and $\oplus$ be binary operators. We say the operator $\odot$ is

left-distributive over $\oplus$ if
$$a \odot (b \oplus c) = a \odot b \oplus a \odot c \qquad \text{for all } a, b, c$$

right-distributive over $\oplus$ if
$$(b \oplus c) \odot a = b \odot a \oplus c \odot a \qquad \text{for all } a, b, c$$

distributive over $\oplus$ if it is left- and right-distributive over $\oplus$. I.e. the operator fulfills the law of *distributivity* over $\oplus$.

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Remark: If the binary operator $\odot$ is commutative the three definitions for distributivity are equivalent.

Remark: For this section the three arguments a, b and c of the last three definitions represent logic statements which are true or false. However, depending on the operators under investigation the arguments may be elements of other sets like the set of real numbers $\mathbb{R}$. We will learn more about sets further down in this chapter.

Theorem 1.7 (Laws on logic)**.** Let a, b and c be statements, then we have:

1. Associativity:
$$(a \wedge b) \wedge c = a \wedge (b \wedge c)$$
$$(a \vee b) \vee c = a \vee (b \vee c)$$

2. Commutativity:
$$a \wedge b = b \wedge a$$
$$a \vee b = b \vee a$$

3. Distributivity over $\wedge$ and $\vee$:
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

4. Double negation:
$$\overline{\overline{a}} = a;$$

5. De Morgan's laws:
$$\overline{a \wedge b} = \overline{a} \vee \overline{b}$$
$$\overline{a \vee b} = \overline{a} \wedge \overline{b}$$

6. Implication and indirect proof:
$$a \Rightarrow b = \overline{a \wedge \overline{b}}$$

(See next remark for an explanation.)

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Proof. All listed laws may be proved by truth tables. Exemplary we prove the first law of De Morgan:

a	b	$\overline{a}$	$\overline{b}$	$a \wedge b$	$\overline{a \wedge b}$	$\overline{a} \vee \overline{b}$
0	0	1	1	0	1	1
0	1	1	0	0	1	1
1	0	0	1	0	1	1
1	1	0	0	1	0	0

The last two columns are equal, hence, the first law of De Morgan holds. □

Remark: The *implication* is often used for theorems: "If statement a holds then statement b holds too." For a false statement a the implication is true for any statement b. Hence, to prove such theorems we focus on a true statement a and look at statement b.

A common way to prove a theorem based on an implication is to perform an *indirect proof*. I.e. we assume that statement a is true and statement b is false. If we then find out that this assumption leads to a contradiction then the original theorem is true.

a	b	$a \Rightarrow b$	$a \wedge \overline{b}$	$\overline{a \wedge \overline{b}}$
0	0	1	0	1
0	1	1	0	1
1	0	0	1	0
1	1	1	0	1

Example 1.4. Indirect proof.

Theorem: $p \in \mathbb{Q} \Rightarrow p^2 \neq 2$, i.e. if p is a rational number then p^2 can not be 2. (I.e. the square root of two can not be expressed by a fraction of integers: $\sqrt{2} \notin \mathbb{Q}$)

Here statement a is $p \in \mathbb{Q}$ and statement b is $p^2 \neq 2$. Instead of proving that $a \Rightarrow b$ (or $\overline{a \wedge \overline{b}}$) is true we prove that $a \wedge \overline{b}$ is false.

Proof: Suppose there is a rational number p and that $p^2 = 2$. Since for statement b the sign of p plays no role we focus on positive rational numbers only.

p being rational means there are natural numbers $m, n \in \mathbb{N}$ such that $p = \frac{m}{n}$ and we can assume that m and n are not both even. (Otherwise we can cancel all common 2's.) Then we have

$$p^2 = \frac{m^2}{n^2} = 2 \qquad \text{and thus} \qquad m^2 = 2n^2$$

Hence m^2 is even. If m^2 is even then m is also even. (The square of an odd number is again odd.) With m even we can write for $k \in \mathbb{N}$:

$$m^2 = (2k)^2 = 4k^2 = 2n^2 \quad \text{and thus} \quad n^2 = 2k^2$$

Hence n^2 and subsequently n is even. I.e. m and n are both even which is a contradiction to our assumption. Hence, since $(p \in \mathbb{Q}) \wedge (p^2 = 2)$ is false we find $p \in \mathbb{Q} \Rightarrow p^2 \neq 2$ is true. ◁

Definition 1.8 (Quantifiers)**.** In addition to the above specified logical operators with one or two arguments we have the so-called *quantifiers* with a variable number of arguments:
universal quantifier:
$\forall x$: "for all x"
existential quantifier:
$\exists x$: "there is (exists) an x" ◁

Example 1.5.

1. $\forall x : x+1 = 1+x, \quad x \in \mathbb{N}$
(Commutativity of addition for any natural number with 1.)

2. $\forall x\ \exists y : y > x, \quad x, y \in \mathbb{N}$
(There is always a larger natural number.)

3. $\forall x : x \neq \sqrt{2}, \quad x \in \mathbb{Q}$
($\sqrt{2}$ is an irrational number.)

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Example 1.6. We apply De Morgan's laws on an arbitrary number of statements x_k, $k \in \mathbb{N}$:

- $\overline{\forall x_k} \;\Leftrightarrow\; \exists\ \overline{x_k}$
E.g. the statement "Not all apples taste well" equals the statement "There exists an apple that does not taste well."

- $\overline{\exists x_k} \;\Leftrightarrow\; \forall\ \overline{x_k}$
E.g. the statement "There does not exists an elephant that can fly" equals the statement "All elephants cannot fly."

◁

1.2. Sets

Definition 1.9 (Sets)**.** Georg Cantor (1845 - 1918):

> A *set* is "*a collection of certain well defined objects of our perception or thinking.*"

The objects contained in a set are called *elements* of the set. For an element x and a set M we write

$$x \in M \quad \text{if } x \text{ is an element of } M$$
$$x \notin M \quad \text{if } x \text{ is not an element of } M$$

The sequential arrangement of the elements does not change the set. Every element appears only once in a set. $\lhd$

Example 1.7.
$\{a, b, c, d\}$ is a set.
$\{a, b, c, c\}$ is not a set.
$\{a, b, c\} = \{a, c, b\} = \{b, a, c\} = \{b, c, a\}$ etc.
$\lhd$

Remark: In order to define a set M, there are two different ways:

1. Extensional definition: We itemize all elements of the set.
$$M = \{a_1, a_2, a_3, \ldots\}$$
2. Intensional definition: We define the set by its characteristic properties.
$$M = \{x \mid x \text{ with properties } E_1, E_2, E_3, \ldots\}$$

Example 1.8. Extensional definition of sets:

$M_1 = \{0, 1\}$ (set of boolean values)
$M_2 = \{\clubsuit, \spadesuit, \heartsuit, \diamondsuit\}$ (colours of playing cards)
$M_3 = \{a, b, c, \ldots, z\}$ (lower case alphabet)
$M_4 = \{2, 4, 6, 8, \ldots\}$ (positive even numbers)
$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (set of integers)

With these sets we have:

$$\begin{array}{rr} 0 \in M_1, & 2 \notin M_1, \\ \heartsuit \in M_2, & 1 \notin M_2, \\ a \in M_3, & A \notin M_3, \\ 4 \in M_4, & 5 \notin M_4, \\ 1 \in \mathbb{Z}, & 1.5 \notin \mathbb{Z}. \end{array}$$

$\lhd$

Example 1.9. Intensional definition of sets:

$M_1 = \{x \in \mathbb{Z} \mid x = 2k, k \in \mathbb{Z}\}$ (even numbers)
$M_2 = \{x \in \mathbb{N} \mid x = k^2, k \in \mathbb{N}\}$ (square numb.)
$M_3 = \{x \in \mathbb{R} \mid x^2 - 4x + 3 = 0\}$
$M_4 = \{x \in \mathbb{Q} \mid |x| \leq 2\}$

With these sets we have:

$$\begin{array}{rr} -4 \in M_1, & 3 \notin M_1, \\ 9 \in M_2, & -9 \notin M_2, \\ 1 \in M_3, & 2 \notin M_3, \\ 2 \in M_4, & \sqrt{2} \notin M_4. \end{array}$$

$\lhd$

Remark: The term $x \in M$ has a logical result which is either true or false and we have:

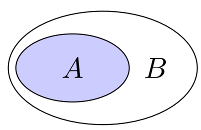
$$x \in M = \overline{x \notin M} \qquad \text{and} \qquad x \notin M = \overline{x \in M}$$

I.e. an element x either belongs to a set M or not — there is nothing intermediate.

Definition 1.10 (Subsets, equal- and empty sets)**.** Let A and B be sets.

- We say A is a *subset* of B if every element of A is also an element of B:
$$A \subseteq B \quad \Leftrightarrow \quad \forall x : x \in A \Rightarrow x \in B$$
- We say A *equals* B if every element of A is also an element of B and vice versa:
$$\begin{array}{rl} & A = B \quad \Leftrightarrow \quad \forall x : x \in A \Leftrightarrow x \in B \\ \text{or:} & A = B \quad \Leftrightarrow \quad A \subseteq B \wedge B \subseteq A \end{array}$$
(Obviously, $A \neq B$ means $\overline{A = B}$)
- We say A is a *proper subset* of B if A is a subset of B but A does not equal B:
$$A \subset B \quad \Leftrightarrow \quad A \subseteq B \wedge A \neq B$$
- We say A is an *empty set* if it contains no elements:
$$\emptyset = \{\ \}$$

$\lhd$



$A \subset B$

Example 1.10. With $A = \{1, 2, 3\}$, $B = \{1, 2, 3\}$ and $C = \{1, 2, 3, 4\}$ we have

$$A \subseteq B, \qquad A = B, \qquad A \subset C$$

An empty set:

$$\{x \in \mathbb{R} \mid x^2 + 1 = 0\} = \emptyset$$

$\lhd$

Definition 1.11 (Union, intersection, complement)**.** With A and B being sets and U the set of all possible elements we define the following operators:

$$
\begin{aligned}
A \cup B &= \{x \mid x \in A \vee x \in B\} && \text{(union)} \\
A \cap B &= \{x \mid x \in A \wedge x \in B\} && \text{(intersection)} \\
A \setminus B &= \{x \mid x \in A \wedge x \notin B\} && \text{(complement)} \\
A^C = \overline{A} &= \{x \mid x \in U \wedge x \notin A\} && \text{(absolute compl.)}
\end{aligned}
$$

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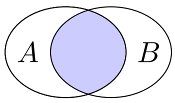
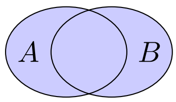
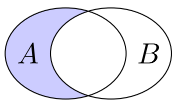


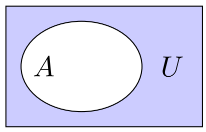
$A \cap B$



$A \cup B$



$A \setminus B$



$\overline{A}$

Remark: For $A \setminus B$ read "complement of A to B". The absolute complement equals the complement of U to A:

$$\overline{A} = U \setminus A$$

Theorem 1.12 (Laws on sets)**.** Let A, B and C be sets, then we have:

1. Associativity:

$$
\begin{aligned}
(A \cap B) \cap C &= A \cap (B \cap C) \\
(A \cup B) \cup C &= A \cup (B \cup C)
\end{aligned}
$$

2. Commutativity:

$$
\begin{aligned}
A \cap B &= B \cap A \\
A \cup B &= B \cup A
\end{aligned}
$$

3. Distributivity over $\cap$ and $\cup$:

$$
\begin{aligned}
A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\
A \cup (B \cap C) &= (A \cup B) \cap (A \cup C)
\end{aligned}
$$

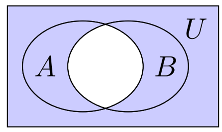
4. Double absolute complement:

$$\overline{\overline{A}} = A$$

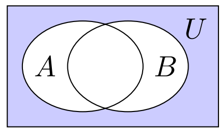
5. De Morgan's laws:

$$
\begin{aligned}
\overline{A \cap B} &= \overline{A} \cup \overline{B} \\
\overline{A \cup B} &= \overline{A} \cap \overline{B}
\end{aligned}
$$

◁



$\overline{A \cap B} = \overline{A} \cup \overline{B}$



$\overline{A \cup B} = \overline{A} \cap \overline{B}$

Proof. The proof is performed by the elements of the sets. Exemplary we prove the first law of De Morgan for any $x \in U$.

$$
\begin{aligned}
x \in \overline{A \cap B} \Leftrightarrow x \notin (A \cap B) &\Leftrightarrow \overline{x \in (A \cap B)} \\
&\Leftrightarrow \overline{(x \in A) \wedge (x \in B)} \\
&\Leftrightarrow \overline{x \in A} \vee \overline{x \in B} \\
&\Leftrightarrow (x \notin A) \vee (x \notin B) \\
&\Leftrightarrow (x \in \overline{A}) \vee (x \in \overline{B}) \\
&\Leftrightarrow x \in (\overline{A} \cup \overline{B})
\end{aligned}
$$

□

1.3. Functions

Definition 1.13 (Cartesian product)**.**
Let $M_1, M_2, \ldots, M_n$ be sets, then the set of all n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_i \in M_i$ for $i = 1, \ldots, n$ is called the *Cartesian product* of $M_1, M_2, \ldots, M_n$ and is denoted by $M_1 \times M_2 \times \ldots \times M_n$, i.e.

$$
\begin{aligned}
&M_1 \times M_2 \times \ldots \times M_n \\
&\quad = \{(a_1, a_2, \ldots, a_n) \mid a_i \in M_i, i = 1, \ldots, n\}
\end{aligned}
$$

◁

Remark: The tuple-elements of a Cartesian product are ordered, e.g. for an Cartesian product of two sets we have:

$$(a_1, a_2) \neq (a_2, a_1)$$

I.e. for two sets A and B we have

$$A \times B \neq B \times A$$

unless A and B are equal.

Example 1.11. Each card of the 52 cards of a standard playing card set (excluding jokers) has a rank (2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king or ace) and a colour (♣, ♠, ♡, ♢).

With rank

$$R = \{2, 3, 4, 5, 6, 7, 8, 9, 10, \mathrm{J}, \mathrm{Q}, \mathrm{K}, \mathrm{A}\}$$

and colour

$$C = \{\clubsuit, \spadesuit, \heartsuit, \diamondsuit\}$$

we may look at the playing cards P as the Cartesian product of R and C:

$$\begin{aligned} P = R \times C = &\{(2, \clubsuit), (2, \spadesuit), (2, \heartsuit), (2, \diamondsuit), \\ &(3, \clubsuit), (3, \spadesuit), \dots, (A, \heartsuit), (A, \diamondsuit)\} \end{aligned}$$

◁

Example 1.12. All possible positions in a three dimensional space $\mathbb{R}^3$ may be described by the Cartesian product of three sets of real numbers $\mathbb{R}$:

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x_1, x_2, x_3) \mid x_i \in \mathbb{R}\}$$ ◁

Definition 1.14 (Binary relation)**.** With A and B being sets we call

$$R \subseteq A \times B$$

a *binary relation*. ◁

As the name *binary relation* suggests it describes the *relation* between two sets only.

Example 1.13. Let x and y be natural numbers. The term $x \geq y$ may be looked at as a binary relation R which is a subset of $\mathbb{N} \times \mathbb{N}$:

$$\begin{aligned} R &= \{(x, y) \mid x \geq y, x \in \mathbb{N}, y \in \mathbb{N}\} \\ &= \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3), \dots\} \\ &\subset \mathbb{N} \times \mathbb{N} \end{aligned}$$

and we get

$$(x, y) \in R \quad \Leftrightarrow \quad x \geq y$$

x →

y ↓

	1	2	3	4	5	6	7 ···
1	√	√	√	√	√	√	√
2		√	√	√	√	√	√
3			√	√	√	√	√
4				√	√	√	√
5					√	√	√
6						√	√
7							√
⋮							

◁

Definition 1.15 (Function)**.** Let A, B be sets, $R \subseteq A \times B$ be a binary relation and let R be such that

1. for all $x \in A$ there is a $y \in B$ such that $(x, y) \in R$
2. if $(x, y_1), (x, y_2) \in R$ then $y_1 = y_2$.

If 1 and 2 are satisfied, we can interpret R as a *function* $f : A \to B$.

We call A the *domain* and B the *codomain*. Instead of writing $(x, y) \in R$ we write $x \mapsto f(x)$. ◁

Example 1.14.

1. Square of natural numbers:
$$f : \begin{cases} \mathbb{N} \to \mathbb{N} \\ x \mapsto x^2 \end{cases}$$
2. Cosine of real numbers:
$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ \varphi \mapsto \cos \varphi \end{cases}$$
3. Natural logarithm:
$$f : \begin{cases} \mathbb{R}_{>0} \to \mathbb{R} \\ x \mapsto \ln x \end{cases}$$

◁

Remark: Every element of the domain is covered exactly once by the function. I.e. every element of the domain is mapped by one tupel of the function to one element of the codomain.

On the other hand, not all elements of the codomain must be mapped to by the function and some elements of the codomain may be mapped to more than once.

The input of a function, i.e. an element of the domain, is called *argument* of the function. The output of a function, i.e. and element of the codomain, is called *value* of the function.

Definition 1.16 (Identity and constant function)**.**

1. Let M be a set and f a function. We say f is an *identity function* id_M if

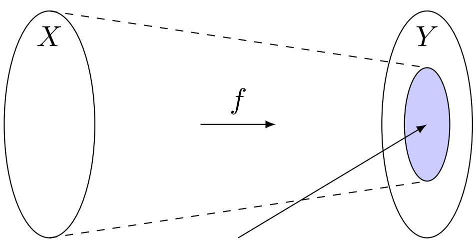
$$f : \begin{cases} M \to M \\ x \mapsto x \end{cases}$$

2. Let X and Y be sets, $y_0 \in Y$ and f be a function. We say f is a *constant function* if

$$f : \begin{cases} X \to Y \\ x \mapsto y_0 \end{cases}$$

◁

Definition 1.17 (Image of a function)**.** Let X and Y be sets and $f : X \to Y$ a function. The set of all possible values of function f is called *image of function* f.



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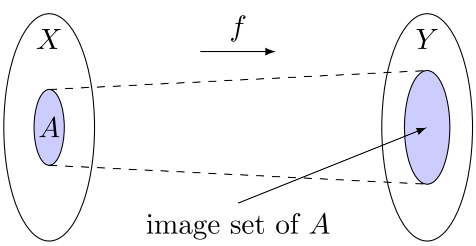
Definition 1.18 (Image and inverse image set)**.** Let X and Y be sets and $f : X \to Y$ a function. For the subsets $A \subseteq X$ and $B \subseteq Y$ we define:

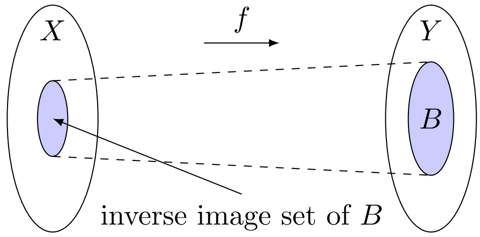
1. *image set* of A:

$$f(A) = \{f(x) \mid x \in A\} \subseteq Y$$



2. *inverse image set* of B:

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subseteq X$$



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Remark: By this definition the function f is used in two different ways:

1. The function f maps an element x of its domain to an element y of its codomain.

$$x \mapsto y = f(x)$$

2. At the same time the function f maps a set X to the corresponding image set Y:

$$X \to Y = f(X)$$

Example 1.15. Let f be a function:

$$f : \begin{cases} \mathbb{Z} \to \mathbb{Z} \\ x \mapsto x^2 \end{cases}$$

then

$$\begin{aligned} f(\{1,2,3\}) &= \{1,4,9\} \\ f^{-1}(\{1,4,9\}) &= \{-3,-2,-1,1,2,3\} \\ f^{-1}(\{2,3\}) &= \{\} = \emptyset \end{aligned}$$

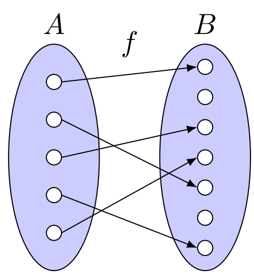
◁

Definition 1.19 (Injective, surjective and bijective functions)**.** Let A and B be sets and f be a function:

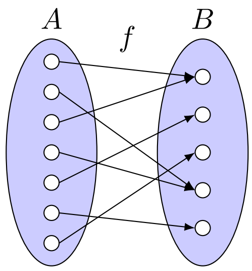
$$f : \begin{cases} A \to B \\ x \mapsto f(x) \end{cases}$$

1. f is called *injective* (or *one-to-one*) if

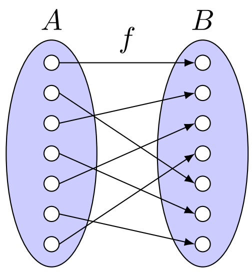
   $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.



2. f is called *surjective* (or *on-to*) if for all $y \in B$ there are one or more $x \in A$ such that $f(x) = y$.



3. f is called *bijective* if it is injective and surjective (*one-to-one* and *on-to*).



$\lhd$

Example 1.16. Let $f : \begin{cases} \mathbb{N} \to \mathbb{N} \\ x \mapsto 2 \cdot x \end{cases}$.
f is *injective*: $2x_1 = 2x_2 \Rightarrow x_1 = x_2$
f is not *surjective*: e.g. 3 is not an image
f is not *bijective*: since f is not surjective. $\lhd$

Example 1.17.
Let $f : \begin{cases} \mathbb{N} \to \{0, 1\} \\ x \text{ even} \mapsto 0, x \text{ odd} \mapsto 1 \end{cases}$
f is not *injective*: e.g. $f(1) = f(3) \Rightarrow 1 \neq 3$
f is *surjective*: 0 and 1 are images
f is not *bijective*: since f is not injective. $\lhd$

Example 1.18. Let $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^2 \end{cases}$
f is not *injective*: e.g. $f(1) = f(-1) \Rightarrow 1 \neq -1$
f is not *surjective*: e.g. -1 is not an image
f is not *bijective*: since f is not injective and not surjective. $\lhd$

Example 1.19. Let $f : \begin{cases} \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \\ x \mapsto x^2 \end{cases}$
f is *injective*: ${x_1}^2 = {x_2}^2 \Rightarrow x_1 = x_2$
f is *surjective*: $f(\mathbb{R}_{\geq 0}) = \mathbb{R}_{\geq 0}$
f is *bijective*: f is injective and surjective. $\lhd$

Definition 1.20 (Inverse function)**.** Let X and Y be sets and f be a bijective function $f : X \to Y$. We then call

$$f^{-1} : \begin{cases} Y \to X \\ f(x) \mapsto x \end{cases}$$

an *inverse function*. $\lhd$

Example 1.20. The function

$$f : \begin{cases} \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \\ x \mapsto x^2 \end{cases}$$

has the inverse function

$$f^{-1} : \begin{cases} \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \\ x^2 \mapsto x \end{cases}$$

which is the square root:

$$x = f^{-1}(y) = \sqrt{y}$$

where $y = x^2$. $\lhd$

1.4. Problems

Problem 1.1: Let A, B and C be statements that are either *true* or *false*. Give the truth table of the following terms:

1. $\overline{A \wedge B}$
2. $\overline{A} \vee \overline{B}$
3. $\overline{A \vee B}$
4. $\overline{A} \wedge \overline{B}$
5. $A \wedge (B \vee C)$
6. $(A \wedge B) \vee (A \wedge C)$
7. $A \vee (B \wedge C)$
8. $(A \vee B) \wedge (A \vee C)$

Problem 1.2: Prove the following tautologies (i.e. terms which are always true):

1. $A \vee \overline{A}$
2. $(A \Rightarrow B) \Leftrightarrow \overline{A \wedge \overline{B}}$
3. $(A \Leftrightarrow B) \Leftrightarrow ((A \wedge B) \vee (\overline{A} \wedge \overline{B}))$

Problem 1.3: Prove the first and second distributive law for conjunction and disjunction.

Problem 1.4: Express the following terms with conjunction and negation only:

1. $A \vee B$
2. $A \Rightarrow B$
3. $A \Leftrightarrow B$

Problem 1.5: Express the following statements with quantifiers:

1. "Every problem has a solution."
2. "Not all people with a driving license own a car."
3. "There are passengers in public transport who do not have a valid ticket."

Problem 1.6: Express the statements of the previous problem with negated fragments in plain words and with quantifiers.

Problem 1.7: Which of the following sets are defined correctly?

1. $M_1 = \{1, 2, 3, 5\}$
2. $M_2 = \{0, 1, 2, a, b, c\}$
3. $M_3 = \{0, 1, \ldots, 9\}$
4. $M_4 = \{0, -1, 1, -2, 2, \ldots\}$
5. $M_5 = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
6. $M_6 = \{\frac{q}{p} \mid q, p \in \mathbb{Z}, p \neq 0\}$
7. $M_7 = \{x \in \mathbb{R} \mid x^2 = 4\}$
8. $M_8 = \{x \in \mathbb{R} \mid x^2 = -1\}$
9. $M_9 = \{x \in \mathbb{Z} \mid x^2 = 2\}$

Problem 1.8: Give an intensional definition of the following sets:

1. $M_1 = \{2, 4, 6, 8, \ldots\}$
2. $M_2 = \{1, 4, 9, 16, \ldots\}$
3. $M_3 = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$
4. $M_4 = \{\frac{2}{1}, \frac{4}{2}, \frac{6}{6}, \frac{8}{24}, \frac{10}{120}, \ldots\}$

Problem 1.9: Give an extensional definition for the following sets:

1. $M_1 = \{x \in \mathbb{N} \mid x = 3n, n \in \mathbb{N}\}$
2. $M_2 = \{x \in \mathbb{N} \mid x = n!, n \in \mathbb{N}\}$
3. $M_3 = \{x \in \mathbb{R} \mid x^2 = 4\}$
4. $M_4 = \{x \in \mathbb{R} \mid x^3 - x^2 - 2x = 0\}$

Problem 1.10: For the following sets A and B find out which terms are correct:

set A	$\{1, 2, 3\}$	$\{a, b, c, d\}$	$\mathbb{N}$	$\mathbb{N}$
set B	$\{1, 2, 3, 4\}$	$\{a, b, d\}$	$\mathbb{N}$	$\mathbb{Z}$
$A = B$				
$A \neq B$				
$A \subseteq B$				
$A \subset B$				
$A \supseteq B$				
$A \supset B$				

Problem 1.11: With $A = \{1, 2, 3, a, b, c\}$, $B = \{1, 2, 3, 4, 5\}$ and $C = \{a, b, c, d, e\}$ give the result of the following terms by extensional definition:

1. $A \cup B$
2. $A \cap B$
3. $A \cap C$
4. $A \setminus B$
5. $(B \cup C) \setminus A$

Problem 1.12: With $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$ write the Cartesian product $A \times B$ by extensional definition.

Problem 1.13: With $A = \{0, 1\}$ find the Cartesian product $A \times A \times A$ in extensional form.

Problem 1.14: Write the following functions as a subset of the Cartesian product by extensional definition:

1. $f : \begin{cases} \mathbb{N} \to \mathbb{N} \\ x \mapsto 2x + 1 \end{cases}$
2. $f : \begin{cases} \mathbb{N} \to \mathbb{Z} \\ x \mapsto x^2 \end{cases}$

Problem 1.15: With $A = \{1, 2, 3\}$ and $R \subseteq A \times A$ check for Cartesian product, binary relation and function:

subset $R \subseteq A \times A$	Cartesian product	binary relation	function
$\{(1,1),(2,2),(3,3)\}$			
$\{(1,1),(1,2),(1,3),$ $(2,1),(2,2),(2,3),$ $(3,1),(3,2),(3,3)\}$			
$\{(1,1),(2,1),(2,2),$ $(3,2),(3,3)\}$			
$\{(1,3),(2,2),(3,1)\}$			
$\{(1,2),(2,2),(3,2)\}$			

Problem 1.16: Express the terms *domain* and *codomain* of a function in your own words.

Problem 1.17: Express the terms *image set* and *inverse image set* of a function in your own words.

Problem 1.18: Check the following terms for injectivity, surjectivity and bijectivity:

term	injective	surjective	bijective
$f : \begin{cases} \mathbb{N} \to \mathbb{N} \\ x \mapsto 3x + 1 \end{cases}$			
$f : \begin{cases} \mathbb{Z} \to \mathbb{N} \cup \{0\} \\ x \mapsto x^2 \end{cases}$			
$f : \begin{cases} \mathbb{N} \to \mathbb{Z} \\ x \mapsto x \end{cases}$			
$f : \begin{cases} \mathbb{Z} \to \mathbb{Z} \\ x \mapsto x \end{cases}$			
$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^3 \end{cases}$			
$f : \begin{cases} \mathbb{R} \to \mathbb{R}_{\geq 0} \\ x \mapsto \exp(x) \end{cases}$			
$f : \begin{cases} \mathbb{R}_{>0} \to \mathbb{R} \\ x \mapsto \ln(x) \end{cases}$			

2. Natural numbers and integers

Mathematics is closely related to numbers. In this chapter we start with two sets of numbers: *natural numbers* and *integers*. In the following chapters we proceed with *rational numbers* (fractions) and *real numbers* before we dive into *complex numbers*.

We start with very basic definitions of numbers. However, we assume the reader understands the principle concept of numbers, addition and multiplication. We skip profound questions like: What is a number? What is infinity?

2.1. Natural numbers $\mathbb{N}$

Definition 2.1 (Natural numbers)**.** We call the set of integers larger than zero *natural numbers*

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

with the two operators '+' and '·' for *addition* and *multiplication*, respectively. ◁

Theorem 2.2 (Properties of $\mathbb{N}$)**.** For $a, b, c \in \mathbb{N}$ we have:

1. Closure:
$$a + b \in \mathbb{N}$$
$$a \cdot b \in \mathbb{N}$$
2. Associativity:
$$(a + b) + c = a + (b + c)$$
$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$
3. Commutativity:
$$a + b = b + a$$
$$a \cdot b = b \cdot a$$
4. Distributivity over $+$:
$$a \cdot (b + c) = a \cdot b + a \cdot c$$
5. Neutral element for $\cdot$:
$$a \cdot 1 = a$$

◁

Remark: In *logic* the distributivity holds over both operators, *conjunction* and *disjunction*. For *sets* the distributivity holds over *union* and *intersection*. For natural numbers distributivity holds only for *multiplication* over *addition* and not vice versa.

Definition 2.3 (Divisor)**.** A natural number d is called *divisor* of $n \in \mathbb{N}$ if there exits a natural number q such that

$$n = d \cdot q$$

In short we write $d \mid n$. ◁

Remark: Every natural number n has at least the two *trivial divisors* 1 and n. The other divisors are called *non-trivial divisors*.

Natural numbers divisible by 2 are called *even*. Natural numbers not divisible by 2 are called *odd*.

Definition 2.4 (Multiple)**.** A natural number m is called a *multiple* of $n \in \mathbb{N}$ if there exists a natural number f such that

$$m = f \cdot n$$ ◁

Remark: If m is a multiple of n with $m = f \cdot n$ then f is a divisor of m. Since 1 is a natural number any natural number n is a multiple of itself: $n = 1 \cdot n$.

Definition 2.5 (Prime number)**.** A natural number n larger than 1 is called *prime* if it has only the divisors 1 and n. ◁

Example 2.1. The set of the first ten prime numbers: $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$ ◁

Definition 2.6 (Prime factor)**.** Let n be a natural number. We call $p \in \mathbb{N}$ a *prime factor* of n if it is prime and a divisor of n. ◁

Theorem 2.7 (Unique prime factorization)**.** Any natural number greater than one can be expressed by a unique product of primes. ◁

Example 2.2.

- $21 = 3 \cdot 7$
- $60 = 2 \cdot 2 \cdot 3 \cdot 5$
- $37 = 37$
- $999\,999 = 3 \cdot 3 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$

$\triangleleft$

Definition 2.8 (Product of sequences)**.** For $m, n \in \mathbb{N}$ and $m \leq n$ let a_k be a number for all $k \in \{m, m+1, \ldots, n-1, n\}$. We then write the product of all a_k:

$$\prod_{k=m}^{n} a_k := a_m \cdot a_{m+1} \cdot \ldots \cdot a_{n-1} \cdot a_n$$

For $m > n$ we define:

$$\prod_{k=m}^{n} a_k := 1$$

$\triangleleft$

Remark: The product of sequences may be used to express prime factorization. With $p_1, p_2, \ldots, p_n$ for n prime factors we can write

$$\prod_{k=1}^{n} p_k = p_1 \cdot p_2 \cdot \ldots \cdot p_{n-1} \cdot p_n$$

Definition 2.9 (Greatest common divisor)**.** With $m, n \in \mathbb{N}$ we call $p \in \mathbb{N}$ the *greatest common divisor* $p = \gcd(m, n)$ if

1. p is a common divisor of m and n, i.e.
$$p \mid m \quad \wedge \quad p \mid n$$
2. any other possible common divisor of m and n is smaller than p.

$\triangleleft$

Example 2.3.

- $\gcd(12, 8) = 4$
- $\gcd(9, 162) = 9$
- $\gcd(45, 49) = 1$
- $\gcd(8638, 7404) = 1234$

$\triangleleft$

Remark: The greatest common divisor can be used to cancel down fractions.

Example 2.4.

- $\dfrac{12}{8} = \dfrac{3 \cdot 4}{2 \cdot 4} = \dfrac{3}{2}$
- $\dfrac{9}{162} = \dfrac{1 \cdot 9}{18 \cdot 9} = \dfrac{1}{18}$
- $\dfrac{45}{49} = \dfrac{45 \cdot 1}{49 \cdot 1} = \dfrac{45}{49}$
- $\dfrac{8638}{7404} = \dfrac{7 \cdot 1234}{6 \cdot 1234} = \dfrac{7}{6}$

$\triangleleft$

Remark: The greatest common divisor p of two natural numbers m and n can be derived by the common prime factors. I.e. the prime factors of m and n are listed side by side and the common primes are then multiplied which leads to the greatest common divisor.

Example 2.5. What is the greatest common divisor of 180 and 168?

$$\begin{aligned}
180 &= 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 = 2^2 \cdot 3^2 \cdot 5^1 \cdot 7^0 \\
168 &= 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7 = 2^3 \cdot 3^1 \cdot 5^0 \cdot 7^1 \\
\gcd &= 2^2 \cdot 3^1 \cdot 5^0 \cdot 7^0 = 12
\end{aligned}$$

$\triangleleft$

Definition 2.10 (Least common multiple)**.** Let m and n be natural numbers. We call $p \in \mathbb{N}$ the *least common multiple* $p = \operatorname{lcm}(m, n)$ if

1. p is a common multiple of m and n, i.e. m and n are both divisors of p:
$$m \mid p \quad \wedge \quad n \mid p$$
2. any other common multiple of m and n is greater than p.

$\triangleleft$

Example 2.6.

- $\operatorname{lcm}(12, 8) = 24$
- $\operatorname{lcm}(9, 162) = 162$
- $\operatorname{lcm}(45, 49) = 2\,205$
- $\operatorname{lcm}(8\,638, 7\,404) = 51\,828$

$\triangleleft$

Remark: The least common multiple is useful when adding fractions.

Example 2.7.

$$\frac{1}{12}+\frac{1}{8}=\frac{1\cdot 2}{12\cdot 2}+\frac{1\cdot 3}{8\cdot 3}=\frac{2+3}{24}=\frac{5}{24}$$

$\triangleleft$

Remark: The least common multiple p of two natural numbers m and n can be derived by the product of their prime factors where the common prime factors are only included once.

Example 2.8. What is the least common multiple of 180 and 168?

$$\begin{aligned}180&=2\cdot 2\cdot 3\cdot 3\cdot 5=2^2\cdot 3^2\cdot 5^1\cdot 7^0\\168&=2\cdot 2\cdot 2\cdot 3\cdot 7=2^3\cdot 3^1\cdot 5^0\cdot 7^1\\\mathrm{lcm}&=2^3\cdot 3^2\cdot 5^1\cdot 7^1=2\,520\end{aligned}$$

$\triangleleft$

Theorem 2.11 (Product of gcd and lcm)**.** Let m and n be natural numbers, $\gcd(m,n)$ the greatest common divisor and $\mathrm{lcm}(m,n)$ the least common multiple of m and n. We then have

$$\gcd(m,n)\cdot\mathrm{lcm}(m,n)=m\cdot n$$

$\triangleleft$

Example 2.9. With $m=180$ and $n=168$ from the previous examples we have

$$\begin{aligned}\gcd(m,n)\cdot\mathrm{lcm}(m,n)&=m\cdot n\\12\cdot 2520&=180\cdot 168\\30\,240&=30\,240\end{aligned}$$

$\triangleleft$

Theorem 2.12 (Division with remainder)**.** Let a and b be natural numbers, then there exist unique numbers $q\in\mathbb{N}\cup\{0\}$ and $r\in\{0,1,2,\dots,b-1\}$ with

$$a=q\cdot b+r$$

We call a the *dividend*, b the *divisor*, q the *quotient* and r the *remainder*, i.e.
dividend = quotient · divisor + remainder $\triangleleft$

Example 2.10.

- $10=2\cdot 4+2$
- $15=2\cdot 6+3$
- $149=2\cdot 50+49$
- $128=8\cdot 16+0$

$\triangleleft$

Theorem 2.13 (Euclidean algorithm)**.** For two natural numbers m and n we can find the greatest common divisor $\gcd(m,n)$ by a series of divisions with remainder:

We call the larger of the two numbers a_0 and the smaller a_1. Now we perform a division of a_0 by a_1. If the remainder is zero then a_1 is the gcd, if not we call the remainder a_2 and we divide a_1 by a_2.

This is repeated until there is no remainder left. The last divisor then is the greatest common divisor $\gcd(m,n)$. $\triangleleft$

Example 2.11. Find the greatest common divisor of 276 and 192.

$$\begin{aligned}276&=1\cdot 192+84\\192&=2\cdot 84+24\\84&=3\cdot 24+12\\24&=2\cdot\underbrace{12}_{\text{gcd}}+0\end{aligned}$$

Hence $\gcd(276,192)=12$. $\triangleleft$

2.2. Representation of natural numbers

Yet, when representing a particular natural number we used the decimal number system. However, this is just one of many ways to note numbers.

Example 2.12. *Roman numerals*.
The Romans used a number system with the following characters:

character	value
I	1
V	5
X	10
L	50
C	100
D	500
M	1000
...	...

(The Romans did not have a character for *zero*!) In principle the characters could be placed in any order, however, they are ordered with decreasing values. So, the number 27 is written as:

$$27 = 2 \cdot X + 1 \cdot V + 2 \cdot I = XXVII$$

To avoid four equal characters in a row (e.g. $4 = IIII$) one of the characters is written before the one with the next higher value (e.g. $4 = IV$). Some examples:

$$\begin{aligned} 19 &= \cancel{XVIIII} = XVIV \\ 54 &= LIV \\ 99 &= \cancel{IC} = LXLVIV \\ 1749 &= MDCCXLVIV && (\text{* J. W. Goethe}) \\ 1750 &= MDCCL && (\dagger \text{ J. S. Bach}) \end{aligned}$$

If you try to perform multiplication or other mathematical operations with this number system you will understand why other number system superseded the Roman numerals. ◁

Definition 2.14 (Sum of sequences)**.**
For $m, n \in \mathbb{N}$ and $m \leq n$ let a_k be a number for all $k \in \{m, m+1, \ldots, n-1, n\}$. We then write the sum of all a_k:

$$\sum_{k=m}^{n} a_k := a_m + a_{m+1} + \ldots + a_{n-1} + a_n$$

For $m > n$ we define:

$$\sum_{k=m}^{n} a_k := 0$$ ◁

Example 2.13. Sum of sequences:

$$\begin{aligned} \sum_{k=2}^{5} k^2 &= 4 + 9 + 16 + 25 = 54 \\ \sum_{k=1}^{4} \frac{1}{k} &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} \\ \sum_{k=1}^{\infty} \frac{1}{2^k} &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 1 \\ \sum_{k=5}^{2} k^2 &= 0 \end{aligned}$$

◁

Definition 2.15 (Positional notation)**.** A *positional system* represents a natural number p to a base b with N digits $b_k, k = 0, \ldots, N-1$, each being a natural number $0, 1, \ldots, b-1$ with

$$p = \sum_{k=0}^{N-1} b_k \cdot b^k$$

The digits are written with decreasing order with no spaces between:

$$b_{N-1} b_{N-2} \cdots b_2 b_1 b_0$$

To explicitly state the base of a number the base may by written as a subscript behind the number (the brackets are optional):

$$p = (\ldots b_3 b_2 b_1 b_0)_b$$ ◁

Example 2.14. Below the numbers 0 to 20 in some number systems: decimal (base 10), binary (base 2), quinary (base 5), octal (base 8) and hexadecimal (base 16).

decimal	binary	quinary	octal	hexadecimal
0	0	0	0	0
1	1	1	1	1
2	10	2	2	2
3	11	3	3	3
4	100	4	4	4
5	101	10	5	5
6	110	11	6	6
7	111	12	7	7
8	1000	13	10	8
9	1001	14	11	9
10	1010	20	12	A
11	1011	21	13	B
12	1100	22	14	C
13	1101	23	15	D
14	1110	24	16	E
15	1111	30	17	F
16	10000	31	20	10
17	10001	32	21	11
18	10010	33	22	12
19	10011	34	23	13
20	10100	40	24	14

Except for the quinary all other number systems play an important role in mathematics and computer science. ◁

Example 2.15. Conversion from base b to base 10. We apply the sum of definition 2.15, e.g.

- $101_2 = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 5_{10}$
- $123_8 = 1 \cdot 8^2 + 2 \cdot 8^1 + 3 \cdot 8^0 = 83_{10}$
- $6F_{16} = 6 \cdot 16^1 + 15 \cdot 16^0 = 111_{10}$

◁

Example 2.16. Conversion from base 10 to base b. We repeatedly perform a division with remainder by b until we get the quotient zero. The remainder-values give the number with base b in reverse order.

- $1\,234_{10} \to$ hexadecimal:

$$\left.\begin{array}{rl} 1\,234 = & 77 \cdot 16 + 2 \\ 77 = & 4 \cdot 16 + 13\ (\mathrm{D}) \\ 4 = & 0 \cdot 16 + 4 \end{array}\right\uparrow = 4\mathrm{D}2_{16}$$

- $25_{10} \to$ binary:

$$\left.\begin{array}{rl} 25 = & 12 \cdot 2 + 1 \\ 12 = & 6 \cdot 2 + 0 \\ 6 = & 3 \cdot 2 + 0 \\ 3 = & 1 \cdot 2 + 1 \\ 1 = & 0 \cdot 2 + 1 \end{array}\right\uparrow = 11001_2$$

◁

Example 2.17. Conversion between base 2 and 8. One octal digit gives three binary digits and vice versa. Hence, we only need to memorize the first eight rows of the table in example 2.14 to rapidly convert between the two bases.

- $1010101_2 = 1\ 010\ 101 = 125_8$
- $765_8 = 111\ 110\ 101 = 111110101_2$

◁

Example 2.18. Conversion between base 2 and 16. One hexadecimal digit gives four binary digits and vice versa. Hence, we only need to memorize the first sixteen rows of the table in example 2.14 to rapidly convert between the two bases.

- $111000111000_2 = 1110\ 0011\ 1000 = \mathrm{E}38_{16}$
- $123_{16} = 0001\ 0010\ 0011 = 100100011_2$

Binary numbers are often grouped by four digits which make them more readable. ◁

2.3. Mathematical induction

Example 2.19. Carl Friedrich Gauß being a schoolboy nine years old once was given the task to add the first 100 natural numbers. The teacher hardly finished the sentence when little Carl wrote 5 050 on his tablet. To add the first n numbers he used the following relation:

$$\sum_{k=1}^{n} k = \frac{n \cdot (n+1)}{2}$$

Is the equation correct? Does it hold for any number of summands? How do we prove that this equation holds for any natural number n?

Carl Friedrich Gauß (1777-1855)

◁

Definition 2.16 (Definition by induction)**.** Let $n_0 \in \mathbb{N}$ and a_n be an object for all $n \geq n_0$. A way to define all a_n is

1. extensional definition of the first object a_{n_0} or the first few objects.
2. definition of any subsequent object by its predecessors.

We call this *definition by induction.* ◁

Example 2.20. Factorial numbers

1. $a_0 := 1$
2. $a_n := n \cdot a_{n-1}$ ◁

Example 2.21. Fibonacci number

1. $a_0 := 0$, $a_1 := 1$
2. $a_{n+1} := a_n + a_{n-1}$ ◁

Example 2.22. *Heron's method* for $\sqrt{2}$

1. $a_1 := 1$
2. $a_{n+1} := \dfrac{1}{2}\left(a_n + \dfrac{2}{a_n}\right)$

a_n converges for increasing n towards $\sqrt{2}$. ◁

Theorem 2.17 (Proof by mathematical induction)**.** Let n_0 be a natural number and $A(n)$ be a statement for all $n \geq n_0$. To prove $A(n)$ for all $n \geq n_0$ it is enough to show that

1. $A(n_0)$ is true
2. for any $n \geq n_0$ if $A(n)$ is true then $A(n+1)$ is true, i.e. $A(n) \Rightarrow A(n+1)$

We call the first *induction start* and the second *induction step*. ◁

The following theorems are proved by the method of mathematical induction.

Theorem 2.18 (Gaussian sum)**.**

$$\sum_{k=1}^{n} k = \frac{n \cdot (n+1)}{2}$$ ◁

Proof.
1. Induction start (A_1):

$$\begin{aligned}\sum_{k=1}^{1} k &= \frac{1 \cdot (1+1)}{2}\\ 1 &= 1\end{aligned}$$

2. Induction step ($A_n \Rightarrow A_{n+1}$): If

$$\underbrace{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}_{A_n}$$

then

$$\underbrace{\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}}_{A_{n+1}}.$$

Now the proof:

$$\begin{aligned}\sum_{k=1}^{n+1} k &= \sum_{k=1}^{n} k + (n+1)\\ &= \frac{n(n+1)}{2} + (n+1)\\ &= \frac{n(n+1) + 2(n+1)}{2}\\ &= \frac{(n+1)(n+2)}{2}\end{aligned}$$

□

Theorem 2.19.

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$ ◁

Proof.
1. Induction start (A_1):

$$\begin{aligned}\sum_{k=1}^{1} k^2 &= \frac{1(1+1)(2 \cdot 1+1)}{6}\\ 1^2 &= \frac{1 \cdot 2 \cdot 3}{6}\\ 1 &= 1\end{aligned}$$

2. Induction step ($A_n \Rightarrow A_{n+1}$): If

$$A_n : \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

then

$$A_{n+1} : \qquad \sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}.$$

Proof:

$$\begin{aligned}\sum_{k=1}^{n+1} k^2 &= \sum_{k=1}^{n} k^2 + (n+1)^2\\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2\\ &= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}\\ &= \frac{(n+1)(n(2n+1) + 6(n+1))}{6}\\ &= \frac{(n+1)(2n^2+7n+6)}{6}\\ &= \frac{(n+1)(n+2)(2n+3)}{6}\end{aligned}$$

□

Theorem 2.20 (Bernoulli's inequality)**.** For $b \in \mathbb{R}$, $b \geq -1$ and $n \in \mathbb{N}$ we have:

$$(1+b)^n \geq 1 + n \cdot b$$ ◁

Proof. 1. Induction start (A_1):

$$\begin{aligned}(1+b)^1 &\geq 1 + 1 \cdot b\\ 1+b &\geq 1+b\end{aligned}$$

2. Induction step ($A_n \Rightarrow A_{n+1}$): If

$$(1+b)^n \geq 1 + n \cdot b$$

then

$$(1+b)^{n+1} \geq 1 + (n+1)b$$

and we show

$$\begin{aligned}(1+b)^{n+1} = (1+b)^n(1+b) &\geq (1+nb)(1+b)\\ &= 1 + \underbrace{b+nb}_{(n+1)b} + \underbrace{nb^2}_{\geq 0} \geq 1 + (n+1)b\end{aligned}$$

□

Theorem 2.21 (Geometric sum)**.** For $q \in \mathbb{R} \setminus \{0,1\}$ and $n \in \mathbb{N}$ we have

$$\sum_{k=0}^{n-1} q^k = \frac{1-q^n}{1-q} \qquad \triangleleft$$

Proof. 1. Induction start (A_1):

$$\begin{aligned} \sum_{k=0}^{1-1} q^k &= \frac{1-q^1}{1-q} \\ q^0 &= \frac{1-q}{1-q} \\ 1 &= 1 \end{aligned}$$

2. Induction step ($A_n \Rightarrow A_{n+1}$): If

$$\sum_{k=0}^{n-1} q^k = \frac{1-q^n}{1-q}$$

then

$$\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}.$$

and we show

$$\begin{aligned} \sum_{k=0}^{n} q^k &= \sum_{k=0}^{n-1} q^k + q^n = \frac{1-q^n}{1-q} + q^n \\ &= \frac{1-q^n+q^n-q\cdot q^n}{1-q} = \frac{1-q^{n+1}}{1-q} \end{aligned}$$

□

2.4. Integers $\mathbb{Z}$

Sums and products of natural numbers are again natural numbers (closure). In dealing with engineering problems, one frequently has to resolve equations. Unfortunately an equation like

$$n + x = m$$

does not always have a solution $x \in \mathbb{N}$ for arbitrary numbers $m, n \in \mathbb{N}$.

If we want to solve equations of the above type without limitations, we are led to an extended set of numbers: For each (positive) $n \in \mathbb{N}$ a *negative* number $-n$ is provided. In addition to that the number 0 (zero) is introduced. This new number system is called the set of *integers* denoted by $\mathbb{Z}$.

Definition 2.22 (Negative numbers and zero)**.** *Zero*, denoted by 0, is the number that does not change the value when added to any natural number, i.e.

$$a + 0 = a, \quad a \in \mathbb{N}$$

The set of negative numbers $\mathbb{N}'$ is defined by

$$\mathbb{N}' = \{a' \mid a + a' = 0, a \in \mathbb{N}\}$$

◁

Definition 2.23 (Integers)**.** The set of Integers $\mathbb{Z}$ is the union of natural numbers, zero and negative numbers:

$$\mathbb{Z} = \mathbb{N} \cup 0 \cup \mathbb{N}'$$

◁

Theorem 2.24 (Negative numbers)**.**

- If $a \in \mathbb{Z} \setminus \{0\}$ then either $a \in \mathbb{N}$ or $a \in \mathbb{N}'$
- If $a \in \mathbb{N}$ then there exists an $a' \in \mathbb{N}'$ such that $a + a' = 0$

◁

Remark: Obviously $\mathbb{N} \subset \mathbb{Z}$.

Theorem 2.25 (Properties of $\mathbb{Z}$)**.** For $a, b, c \in \mathbb{Z}$ we have:

1. Closure:
$$\begin{aligned} a + b &\in \mathbb{Z} \\ a \cdot b &\in \mathbb{Z} \end{aligned}$$
2. Associativity:
$$\begin{aligned} (a+b)+c &= a+(b+c) \\ (a \cdot b) \cdot c &= a \cdot (b \cdot c) \end{aligned}$$
3. Commutativity:
$$\begin{aligned} a + b &= b + a \\ a \cdot b &= b \cdot a \end{aligned}$$
4. Distributivity over +:
$$a \cdot (b + c) = a \cdot b + a \cdot c$$
5. Neutral element for + and ·:
$$\begin{aligned} a + 0 &= a \\ a \cdot 1 &= a \end{aligned}$$

6. Inverse element for $+$: There exists an inverse element $a' \in \mathbb{Z}$ such that

$$a + a' = 0$$

7. If $a \cdot b = 0$ then either $a = 0$ or $b = 0$:

$$a \cdot b = 0 \quad \Leftrightarrow \quad (a = 0) \vee (b = 0)$$

$\lhd$

Definition 2.26 (Factorial)**.** We define the factorial of n as

$$n! : \begin{cases} \mathbb{N} \cup 0 \to \mathbb{N} \\ n \mapsto \prod_{k=1}^{n} k \end{cases}$$

$\lhd$

Example 2.23. There are $n!$ options to arrange n items in a chain. $\lhd$

Theorem 2.27 (Binomial theorem)**.** For $x, y \in \mathbb{Z}$ and $n \in \mathbb{N}$ we have

$$(x \pm y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} (\pm y)^k$$

with the binomial coefficient $\binom{n}{k}$ as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$\lhd$

Remark: We may apply the same theorem for x, y being rational, real or complex.

The binomial coefficients are elements of *Pascal's triangle*. The left and right edge are filled with ones. All other elements are the sum of the two elements above it.

$$\xrightarrow{k} \qquad n \downarrow$$

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1
\end{array}$$

$$\xrightarrow{k} \qquad n \downarrow$$

$$\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \binom{1}{1} \\
\binom{2}{0} \binom{2}{1} \binom{2}{2} \\
\binom{3}{0} \binom{3}{1} \binom{3}{2} \binom{3}{3} \\
\binom{4}{0} \binom{4}{1} \binom{4}{2} \binom{4}{3} \binom{4}{4} \\
\binom{5}{0} \binom{5}{1} \binom{5}{2} \binom{5}{3} \binom{5}{4} \binom{5}{5} \\
\binom{6}{0} \binom{6}{1} \binom{6}{2} \binom{6}{3} \binom{6}{4} \binom{6}{5} \binom{6}{6}
\end{array}$$

Example 2.24. We want to expand $(a-b)^4$:

$$\begin{aligned}
(a-b)^4 &= \sum_{k=0}^{4} \binom{4}{k} a^{4-k} (-b)^k \\
&= \tbinom{4}{0} a^4 b^0 - \tbinom{4}{1} a^3 b^1 + \tbinom{4}{2} a^2 b^2 \\
&\quad - \tbinom{4}{3} a^1 b^3 + \tbinom{4}{4} a^0 b^4 \\
&= a^4 - 4a^3 b + 6a^2 b^2 - 4ab^3 + b^4
\end{aligned}$$

$\lhd$

Example 2.25. The binomial coefficient $\binom{n}{k}$ gives you the number of options to choose k out n different items.

E.g. for the German lotto you choose six numbers out of 49, i.e. six different numbers between 1 and 49. There are $\binom{49}{6} = 13\,983\,816$ different options to choose these numbers. Hence the chance to win the top price is $\binom{49}{6}^{-1} \approx 7 \times 10^{-8} = 0.000\,007\%$. $\lhd$

Remark: Unfortunately, equations of the form

$$m \cdot x = n$$

have no solution in $\mathbb{Z}$ for arbitrary $m, n \in \mathbb{Z}$ which leads us to the rational numbers $\mathbb{Q}$ in the next section.

Example 2.26. There exists no $x \in \mathbb{Z}$ that satisfies the following equation:

$$2 \cdot x = 3$$

$\lhd$

2.5. Problems

Problem 2.1: In the context of natural numbers explain the following terms:

1. divisor
2. *trivial* and *non-trivial* divisor
3. multiple

4. prime number
5. prime factor

Problem 2.2: Perform a prime factorization on the following numbers:

1. 123
2. 321
3. 30 030
4. 65 536
5. 9 000 000

Problem 2.3: Simplify the following terms:

1. $\prod_{k=1}^{5} 2$
2. $\prod_{k=1}^{4} k$
3. $\prod_{k=1}^{3} k^2$
4. $\frac{1}{2^n} \prod_{k=1}^{n} 2x$
5. $\prod_{k=1}^{n} k - n!$
6. $\frac{\prod_{k=1}^{7} f^k(x)}{\prod_{k=2}^{7} f^k(x)}$

with $f(x) \neq 0$ for all x

Problem 2.4: Evaluate the *greatest common divisor* and *least common multiple* of the following pairs:

a	b	$\gcd(a, b)$	$\operatorname{lcm}(a, b)$
6	8		
30	20		
14	15		
39	117		
123	123		

Problem 2.5: If the greatest common divisor and the least common multiple of two natural numbers a and b are known, is it possible to evaluate a and b?

Problem 2.6: Complete out the following table:

dividend	divisor	quotient	remainder
100	16		
62	7		
156	26		
123	124		
124	123		
1 000	3		

Problem 2.7: Perform the Euclidean algorithm to the following pairs of natural numbers to find the greatest common divisor:

1. 80 and 115
2. 768 and 540
3. 5 471 and 1 193

Problem 2.8: Change the base of the following representations:

1. $100_{10} \rightarrow$ binary
2. $100_{10} \rightarrow$ octal
3. $100_{10} \rightarrow$ decimal
4. $100_{10} \rightarrow$ hexadecimal
5. $100_{2} \rightarrow$ decimal
6. $100_{8} \rightarrow$ decimal
7. $100_{10} \rightarrow$ decimal
8. $100_{16} \rightarrow$ decimal

Problem 2.9: Fill out the following table:

binary	octal	hexadecimal
11 0111		
1010 0101 1100		
	17	
	6543	
		123
		ABC

Problem 2.10: Prove by mathematical induction:

1. $\sum_{k=1}^{n} k^3 = \frac{n^2 \cdot (n+1)^2}{4}$
2. $n^3 + 2n$ has the divisor 3
3. $n! > 2^n$ for $n > 3$
4. $\sum_{k=1}^{n} (2k-1) = n^2$

Problem 2.11: Prove by mathematical induction that there are $n!$ options to arrange n items into a chain.

3. Rational and real numbers

3.1. Rational numbers $\mathbb{Q}$

Definition 3.1 (Rational numbers)**.** We define the set of *rational numbers* $\mathbb{Q}$ as the quotient of any $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, i.e.

$$\mathbb{Q} = \left\{ \frac{p}{q} \;\middle|\; p \in \mathbb{Z}, q \in \mathbb{N} \right\}$$

$\triangleleft$

Example 3.1.

$$\begin{array}{rrr} \frac{1}{2} \in \mathbb{Q} & -\frac{3}{7} \in \mathbb{Q} & \frac{48}{64} = \frac{3}{4} \in \mathbb{Q} \\ \frac{5}{2} = 2\frac{1}{2} \in \mathbb{Q} & 1 \in \mathbb{Q} & 0 \in \mathbb{Q} \\ \sqrt{2} \notin \mathbb{Q} & \pi \notin \mathbb{Q} & e \notin \mathbb{Q} \end{array}$$

$\triangleleft$

Theorem 3.2 (Properties of $\mathbb{Q}$)**.** For any $a, b, c \in \mathbb{Q}$ and $+$ and $\cdot$ as commonly used we have:

1. Closure:
$$a + b \in \mathbb{Q}$$
$$a \cdot b \in \mathbb{Q}$$

2. Associativity:
$$(a + b) + c = a + (b + c)$$
$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

3. Commutativity:
$$a + b = b + a$$
$$a \cdot b = b \cdot a$$

4. Distributivity over $+$:
$$a \cdot (b + c) = a \cdot b + a \cdot c$$

5. Neutral element for $+$ and $\cdot$:
$$a + 0 = a$$
$$a \cdot 1 = a$$

6. Inverse element: There exists the inverse elements $a', b' \in \mathbb{Q}$ such that
$$a + a' = 0$$
$$b \cdot b' = 1 \qquad \text{for } b \neq 0$$
where 0 and 1 denote the neutral elements for $+$ and $\cdot$, respectively.

7. If $a \cdot b = 0$ then either $a = 0$ or $b = 0$ and vice versa, i.e.
$$a \cdot b = 0 \quad \Leftrightarrow \quad (a = 0) \vee (b = 0)$$

$\triangleleft$

Remark: We denote the inverse element for addition with a minus sign.

$$a' = -a$$

In short we write:

$$a + (-b) \quad \Leftrightarrow \quad a - b$$

We denote the inverse element for multiplication with an exponent of -1 or with b in the denominator of a fraction.

$$b' = b^{-1} = \frac{1}{b}$$

Theorem 3.3 ($\mathbb{Q}$ is countable)**.** The rational numbers $\mathbb{Q}$ are countable. $\triangleleft$

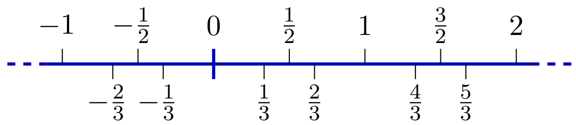
Proof. We arrange the elements of $\mathbb{Q}$ in a table:

$$\begin{array}{cccccc} \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \frac{6}{1} \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \frac{6}{2} \\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \frac{6}{3} \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \frac{6}{4} \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \frac{5}{5} & \frac{6}{5} \\ \frac{1}{6} & \frac{2}{6} & \frac{3}{6} & \frac{4}{6} & \frac{5}{6} & \frac{6}{6} \end{array}$$

and we write:

$\mathbb{Q} = \{0, \pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{2}{1}, \pm\frac{1}{3}, \pm\frac{2}{2}, \pm\frac{3}{1}, \ldots\}$ $\square$

Remark: We can write rational numbers on a real line:



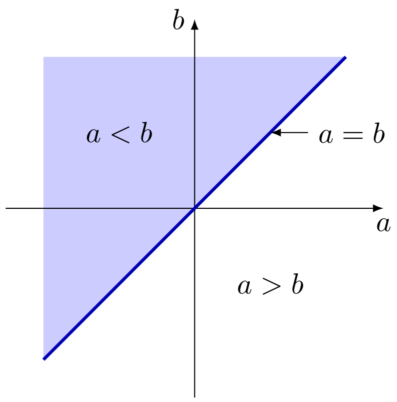
By this visualization each number has a unique position on the line.

Definition 3.4 (Less or equal)**.** For $a, b \in \mathbb{Q}$ we say a is *less or equal* b if a is not to the right of b on the real line. In short we write

$$a \leq b$$

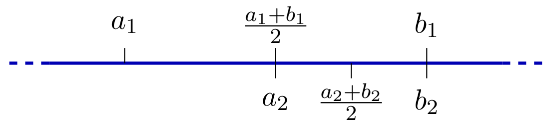
and also call it an *order relation*.

The relations $\geq$, $<$ and $>$ are defined in a similar manner. ◁



Theorem 3.5 ($\mathbb{Q}$ is dense)**.** For every $a, b \in \mathbb{Q}$ with $a < b$ there exists a $c \in \mathbb{Q}$ with $a < c < b$. I.e. for any two unequal rational numbers there exists an infinite number of rational numbers between them. ◁

Proof. We start with two rational numbers $a_1, b_1 \in \mathbb{Q}$ where $a_1 < b_1$. The term $\frac{a_1+b_1}{2}$ is greater than a_1, less than b_1 and again is a rational number. We now either set $a_2 = a_1$ and $b_2 = \frac{a_1+b_1}{2}$ or $a_2 = \frac{a_1+b_1}{2}$ and $b_2 = b_1$. Again the term $\frac{a_2+b_2}{2}$ is greater than a_2, less than b_2 and again is a rational number. This may be repeated infinite times for any position on the real line. $\square$



Remark: Although the rational numbers $\mathbb{Q}$ are dense there exist numbers like $\sqrt{2}$ or π that can not be expressed by rational numbers.

It is possible to approximate such numbers with arbitrary small error larger than zero by an increasing number of digits for numerator and denominator, however, we will never be able to express these numbers exactly.

It is possible to express these *irrational* numbers by an infinite sum of rational numbers. I.e. for π we may write

$$\pi = 4 \cdot \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\right)$$

Hence, we are looking for an extended set of numbers that include irrational numbers.

3.2. Real numbers $\mathbb{R}$

Definition 3.6 (Irrational numbers)**.** Let S be the set of infinite sums of rational numbers. We call the set of all elements $s \in S$ that are not rational the set of *irrational numbers*. ◁

Example 3.2. Some irrational numbers:

- $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots = e$
- $1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots = \frac{1}{e}$
- $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots = \ln 2$
- $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots = \frac{\pi}{4}$

◁

Definition 3.7 (Real numbers)**.** We call the set of all possible sums of rational numbers with finite or infinite number of summands the set of *real numbers* $\mathbb{R}$. ◁

Theorem 3.8 (Properties of $\mathbb{R}$)**.** The set of real numbers $\mathbb{R}$ has the following properties:

- $\mathbb{R}$ is not countable
- $\mathbb{R}$ is dense

◁

3.3. Intervals and absolute values

Definition 3.9 (Finite intervals)**.** For $a, b \in \mathbb{R}$ and $a < b$ we define *finite intervals*:

- *closed interval*:

$$[a, b] = \{x \in \mathbb{R} \mid a \le x \le b\}$$

- *open interval*:

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

- *left open interval*:

$$(a, b] = \{x \in \mathbb{R} \mid a < x \le b\}$$

- *right open interval*:

$$[a, b) = \{x \in \mathbb{R} \mid a \le x < b\}$$

◁

Definition 3.10 (Infinity)**.** We define $+\infty$ (or just ∞) and $-\infty$ such that the statements "$x < \infty$" and "$x > -\infty$" are true for all $x \in \mathbb{R}$. Vice versa the statements "$x \ge \infty$" and "$x \le -\infty$" are false for all $x \in \mathbb{R}$. ◁

Remark: $+\infty$ and $-\infty$ are no real numbers or any other type of numbers. E.g. the terms $\infty - \infty$ or $\frac{\infty}{\infty}$ provide no sensible results!

Definition 3.11 (Infinite intervals)**.** For $a \in \mathbb{R}$ we define *infinite intervals*:

$$\begin{aligned} [a, \infty) &= \{x \in \mathbb{R} \mid a \le x\} \\ (a, \infty) &= \{x \in \mathbb{R} \mid a < x\} \\ (-\infty, a] &= \{x \in \mathbb{R} \mid x \le a\} \\ (-\infty, a) &= \{x \in \mathbb{R} \mid x < a\} \end{aligned}$$

◁

Example 3.3.

- $\mathbb{R} = (-\infty, \infty)$
- $\mathbb{R}_{>0} = (0, \infty) = \{x \in \mathbb{R} \mid x > 0\}$
- $\mathbb{R}_{\ge 0} = [0, \infty) = \{x \in \mathbb{R} \mid x \ge 0\}$
- $\{x \in \mathbb{R} \mid x = \sin(y), y \in \mathbb{R}\} = [-1, 1]$
- $\{x \in \mathbb{R} \mid x = y^2 - 1, 0 \le y < 1\} = [-1, 0)$

◁

Definition 3.12 (Supremum and infimum)**.** Let A be an arbitrary subset of $\mathbb{R}$.

- We call the least number of $\mathbb{R} \cup \{\infty\}$ greater or equal to all elements of A the *supremum* of A or the *least upper bound* of A denoted by
$$\sup(A)$$
- We call the greatest number of $\mathbb{R} \cup \{-\infty\}$ less or equal to all elements of A the *infimum* of A or the *greatest lower bound* of A denoted by
$$\inf(A)$$

◁

Example 3.4.

- $\sup(\{1, 2, 3\}) = 3$
- $\inf(\{1, 2, 3\}) = 1$
- $\sup(\mathbb{R}) = \infty$
- $\inf(\mathbb{R}_{>0}) = 0$
- $\inf(\mathbb{R}_{\ge 0}) = 0$

◁

Definition 3.13 (Maximum and minimum)**.** Let A be an arbitrary subset of $\mathbb{R}$

- If the supremum of A is an element of A we call it the *maximum* of A denoted by
$$\max(A)$$
- If the infimum of A is an element of A we call it the *minimum* of A denoted by
$$\min(A)$$

◁

Example 3.5.

- Let $A = [-1, 1]$ then
$$\max(A) = 1 \quad \text{and} \quad \min(A) = -1$$
- Let $B = \{y \in \mathbb{R} \mid y = x^2, x \in [-1, 1]\}$ then
$$\max(B) = 1 \quad \text{and} \quad \min(B) = 0$$

- Let $C = (0, 10)$ then $\max(C)$ and $\min(C)$ are undefined!

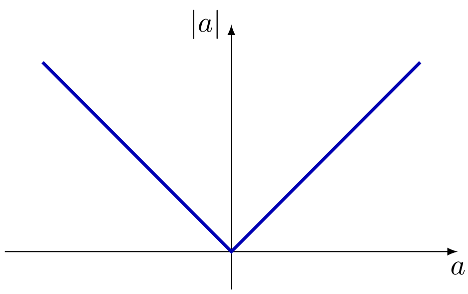
$\triangleleft$

Remark: The minimum and maximum exist for closed intervals and for finite sets (i.e. sets with a finite number of elements). If a maximum or minimum exists it equals the supremum or infimum, respectively.

Definition 3.14 (Absolute value)**.** For $a \in \mathbb{R}$ we define the *absolute value* of a by

$$|a| := \begin{cases} a & \text{for} \quad a \geq 0 \\ -a & \text{for} \quad a < 0 \end{cases}$$

$\triangleleft$



Theorem 3.15 (Properties of absolute values)**.** For any $a, b \in \mathbb{R}$ we have:

- $|a| \geq 0$
- $|a| = 0 \quad \Leftrightarrow \quad a = 0$
- $|a \cdot b| = |a| \cdot |b|$
- $|a + b| \leq |a| + |b|$ (triangle inequality)

$\triangleleft$

3.4. Problems

Problem 3.1: Which of the following is an element of $\mathbb{Q}$?

1.5	$1/2$	π	$\sqrt{3}$	1.234567
-5	$1/3$	e	$\sqrt{4}$	$0.\overline{142857}$

Problem 3.2: With $a = 2$ and $b = 3$ which of the following expressions are true?

a) $a \leq b$	b) $a \geq b$	c) $2a \leq b$
d) $a < b$	e) $a > b$	f) $3a \leq 2b$
g) $2b < \pi a$	h) $a\sqrt{b} \leq b\sqrt{a}$	i) $a\sqrt{b} \geq b\sqrt{a}$

Problem 3.3: Express the term *dense* in your own words.

Problem 3.4: Explain the difference between the sets $\mathbb{Q}$ and $\mathbb{R}$.

Problem 3.5: Give the intervals of the following sets:

1. $A_1 = \{y \in \mathbb{R} \mid y = x^2, x \in \mathbb{R}_{>0}\}$
2. $A_2 = \{y \in \mathbb{R} \mid y = x^2, x \in \mathbb{R}_{\leq 0}\}$
3. $A_3 = \{y \in \mathbb{R} \mid y = x^{-1}, x \in \mathbb{R}_{\geq 1}\}$
4. $A_4 = \{y \in \mathbb{R} \mid y = \exp(x), x \in \mathbb{R}\}$
5. $A_5 = \{y \in \mathbb{R} \mid y = \sin(x), x \in \mathbb{R}\}$

Problem 3.6: Find the supremum and infimum of the following sets.

set	sup	inf
$\{1, 2, 3, 4\}$		
$\{-2, 0, 2, 4\}$		
$\{-100, 200\}$		
$\{3, 6, 9, \ldots\}$		
$\{\ldots, -1, 0, 1, 2, \ldots\}$		

Problem 3.7: For the following sets find the maximum and minimum.

set	max	min
$\{1, 2, 3, 4\}$		
$\{-2, 0, 2, 4\}$		
$\{-100, 200\}$		
$\{3, 6, 9, \ldots\}$		
$\{\ldots, -1, 0, 1, 2, \ldots\}$		

Problem 3.8: Give the extensional definition of the following sets:

1. $A_1 = \{n \in \mathbb{Z} \mid n = |k|, k \in \mathbb{Z}\}$
2. $A_2 = \{y \in \mathbb{R} \mid y = \frac{x}{|x|}, x \in \mathbb{R}_{\neq 0}\}$
3. $A_3 = \{n \in \mathbb{Z} \mid n = |10k - 1|, k \in \mathbb{Z}\}$

4. Complex numbers

4.1. Why complex numbers?

Let us recall the reasons to develop extended number sets.

- We started with natural numbers $\mathbb{N}$.
- With $n \geq m$ the equation $n + x = m$ has no solution for $x \in \mathbb{N}$ which lead us to the integers $\mathbb{Z}$.
- If m is not a multiple of n the equation $n \cdot x = m$ has no solution for $x \in \mathbb{Z}$ which lead us to the rational numbers $\mathbb{Q}$.
- $x^2 - c = 0$ has no solution for $x \in \mathbb{Q}$ if c is not a square of a fraction (e.g. $c = 2$) which lead us to the real numbers $\mathbb{R}$.

The argument which lead to the real numbers $\mathbb{R}$ already point at the next limit to overcome: If c in equation $x^2 - c = 0$ is negative, we get to the point to take the square root of a negative number which has no solution in $\mathbb{R}$.

Hence, we are looking for a number system that gives us a solution for equations like

$$x^2 + c = 0 \qquad \text{with} \quad c > 0$$

4.2. Definition of complex numbers

Definition 4.1 (Imaginary unit)**.** We define the constant $\mathrm{j} \notin \mathbb{R}$ by

$$\mathrm{j}^2 = -1$$

and call it *imaginary unit*. A multiple of j is called *imaginary number*. ◁

Remark: Strictly speaking this definition is not precise since there are two solutions for this equation. In mathematics we say it is not *well defined*. However, for our purposes it is enough to imagine j to be the positive root of -1:

$$\mathrm{j} := +\sqrt{-1}$$

With this supplement the previous definition becomes well defined and all further applications of complex numbers are consistent. (We could do the same by defining $\mathrm{j} := -\sqrt{-1}$, but we don't.)

Remark: In mathematics the letter i is used for the imaginary unit instead. However, in electrical engineering the i is *the* symbol for the electrical current and, hence, we use the j as a symbol for the imaginary unit.

Example 4.1. We want to solve the equation $x^2 = -4$ for x:

$$\begin{aligned} x^2 &= -4 \\ x &= \pm\sqrt{-4} \\ x &= \pm\sqrt{4} \cdot \sqrt{-1} \\ x &= \pm 2 \cdot \mathrm{j} \\ x &= \pm 2\,\mathrm{j} \end{aligned}$$

Towards the end of this chapter we will see that care must be taken when dealing with roots on complex numbers. However, the result $\pm 2\mathrm{j}$ in this example is correct. ◁

Theorem 4.2 (Powers of j)**.** Although j is not real, we may use it as a unit and apply mathematical operations as usual. Powers of j may be combined:

$$\begin{aligned} \mathrm{j}^1 &= \mathrm{j} \\ \mathrm{j}^2 &= -1 \qquad \text{(by definition)} \\ \mathrm{j}^3 &= (\mathrm{j}^2) \cdot \mathrm{j} = (-1) \cdot \mathrm{j} = -\mathrm{j} \\ \mathrm{j}^4 &= \mathrm{j}^2 \cdot \mathrm{j}^2 = (-1) \cdot (-1) = 1 \\ \mathrm{j}^5 &= \mathrm{j}^4 \cdot \mathrm{j} = \mathrm{j} \\ \mathrm{j}^0 &= 1 \\ \mathrm{j}^{-1} &= \tfrac{1}{\mathrm{j}} = \tfrac{\mathrm{j}^4}{\mathrm{j}} = \mathrm{j}^3 = -\mathrm{j} \end{aligned}$$

n	...	-2	-1	0	1	2	3	4	5	...
j^n	...	-1	$-\mathrm{j}$	1	j	-1	$-\mathrm{j}$	1	j	...

◁

Definition 4.3 (Complex number)**.** With $a, b \in \mathbb{R}$ we define complex numbers as $a + \mathrm{j} \cdot b$ or $a + \mathrm{j}\,b$, i.e.

$$\mathbb{C} = \{a + \mathrm{j}\,b \mid a, b \in \mathbb{R}, \mathrm{j}^2 = -1\}$$

We say two complex numbers are equal if their components a and b are equal, i.e. for $c_1 = a_1 + \mathrm{j}\, b_1$ and $c_2 = a_2 + \mathrm{j}\, b_2$ we have:

$$c_1 = c_2 \quad \Leftrightarrow \quad a_1 = a_2 \wedge b_1 = b_2$$

◁

Definition 4.4 (Real and imaginary part)**.** For any complex number $c = a + \mathrm{j}\, b \in \mathbb{C}$ we define

$$\mathrm{Re}(c) = \mathrm{Re}(a + \mathrm{j}\, b) = a$$

as the *real part* of c and

$$\mathrm{Im}(c) = \mathrm{Im}(a + \mathrm{j}\, b) = b$$

as the *imaginary part* of c. ◁

Example 4.2.

- $\mathrm{Re}(3 - \mathrm{j}\, 4) = 3$
- $\mathrm{Im}(3 - \mathrm{j}\, 4) = -4$
- $\mathrm{Re}(\mathrm{j}) = 0$
- $\mathrm{Im}(\mathrm{j}) = 1$
- $\mathrm{Re}(\mathrm{j}^2) = -1$
- $\mathrm{Im}(5) = 0$

◁

Remark: Note that the imaginary part of a complex number is real, i.e. for any $c \in \mathbb{C}$ we have:

$$\mathrm{Im}(c) \in \mathbb{R}$$

As part of a complex number we multiply the imaginary part with j which gives us an imaginary number.

There is another more profound way to define complex numbers:

Definition 4.5 (Complex number, alternative definition)**.** The Cartesian product $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ together with the two operators

$$+ : \begin{cases} \mathbb{C} \times \mathbb{C} \to \mathbb{C} \\ (a_1, b_1) + (a_2, b_2) \mapsto \\ \qquad (a_1 + a_2, b_1 + b_2) \end{cases}$$

$$\cdot : \begin{cases} \mathbb{C} \times \mathbb{C} \to \mathbb{C} \\ (a_1, b_1) \cdot (a_2, b_2) \mapsto \\ \qquad (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1) \end{cases}$$

is called the set of *complex numbers*. ◁

For tuples with the second element being zero we get:

$$(a_1, 0) + (a_2, 0) = (a_1 + a_2, 0)$$
$$(a_1, 0) \cdot (a_2, 0) = (a_1 \cdot a_2, 0)$$

Hence, the first tuple element behaves like the real part of a complex number.

If we square $(0, 1)$ we get:

$$(0, 1) \cdot (0, 1) = (-1, 0)$$

I.e. the tuple (0,1) behaves like the imaginary unit. In fact the second tuple element is the imaginary part of our complex number defined in the first place.

The main advantage of this second definition of complex numbers is that $\mathrm{j} := (0, 1)$ is well defined and needs no further explanation. This is why the second definition is found in more mathematical focused text books whereas the first has a more empirical view point.

4.3. Properties of complex numbers

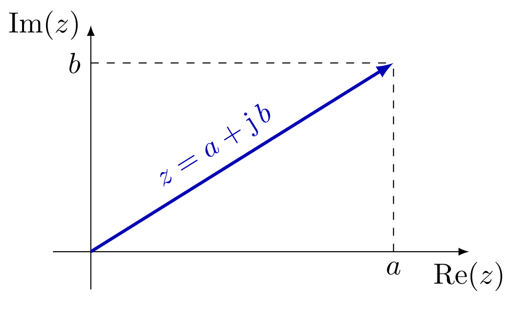
Theorem 4.6 (Properties of $\mathbb{C}$)**.**

- $\mathbb{C}$ is not countable
- $\mathbb{C}$ is dense

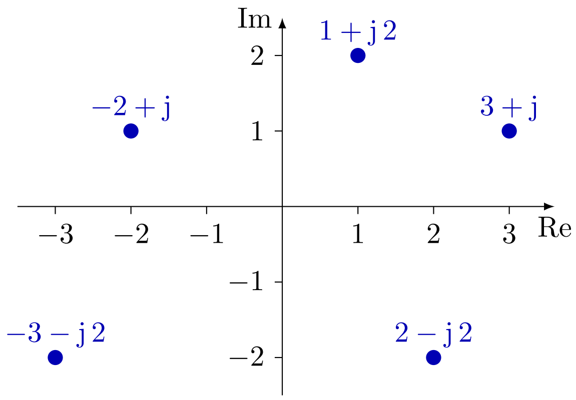
◁

Remark: Another interesting property of complex numbers is that they cannot be ordered. I.e. for two complex numbers $z_1, z_2 \in \mathbb{C}$ the expression $z_1 < z_2$ does not give a clear result. E.g. is $1 - \mathrm{j}$ less or greater than $\mathrm{j} - 1$? Mathematically speaking $\mathbb{C}$ is not a *totally ordered set*. This term is defined by some axioms but is not further discussed here.

Remark: A complex number $z = a + \mathrm{j}\, b$ can be represented as a point (a, b) or an arrow to this point on a *complex plane*. The real and imaginary parts are plotted on the abscissa and ordinate, respectively.



Example 4.3.



◁

Definition 4.7 (Absolute)**.** The *absolute* of a complex number $z = a + \mathrm{j}\, b \in \mathbb{C}$ is defined by

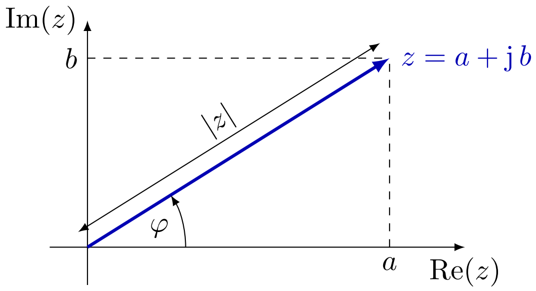
$$|z| := \sqrt{a^2 + b^2}$$

◁

Theorem 4.8 (Polar form)**.** A complex number $z = a + \mathrm{j}\, b \in \mathbb{C}$ can also be expressed by polar coordinates $|z|$ and φ where $|z|$ is the absolute of z and φ the angle from the positive real axis counter-clockwise to the arrow on the complex plane. We have:

$$\begin{aligned} a &= |z| \cdot \cos\varphi \\ b &= |z| \cdot \sin\varphi \\ |z| &= \sqrt{a^2 + b^2} \\ \varphi &= \arg z \end{aligned}$$

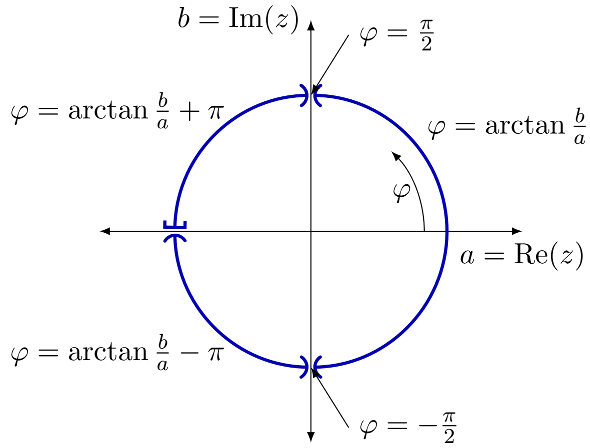
where $\arg z$ is defined as follows. ◁



Definition 4.9 (Argument)**.** With $z = a + \mathrm{j}b$, $a, b \in \mathbb{R}$, $|z| \neq 0$ the function $\arg(z)$ maps a complex number to the interval $(-\pi, \pi]$ defined with:

$$\arg(z) = \begin{cases} \arctan\frac{b}{a} - \pi & \text{for } a < 0 \text{ and } b < 0 \\ -\frac{\pi}{2} & \text{for } a = 0 \text{ and } b < 0 \\ \arctan\frac{b}{a} & \text{for } a > 0 \\ \frac{\pi}{2} & \text{for } a = 0 \text{ and } b > 0 \\ \arctan\frac{b}{a} + \pi & \text{for } a < 0 \text{ and } b \geq 0 \end{cases}$$

◁



Remark: Now a complex number can be written with their polar coordinates.

$$z = a + \mathrm{j}\, b = |z| \cdot (\cos\varphi + \mathrm{j}\, \sin\varphi)$$

Later we will learn Euler's formula:

$$\exp(\mathrm{j}\, x) = e^{\mathrm{j}\, x} = \cos x + \mathrm{j}\, \sin x$$

which further simplifies our complex number to

$$z = a + \mathrm{j}\, b = |z| e^{\mathrm{j}\,\varphi}$$

Remark: By definition 4.9 the polar angle is an element of $(-\pi, \pi]$. Other definitions limit the angle to $[0, 2\pi)$. Due to the periodicity of the exponential function for imaginary values we have:

$$z = |z| \cdot e^{\mathrm{j}\arg(z)} = |z| \cdot e^{\mathrm{j}(2\pi + \arg(z))}$$

or more general:

$$z = |z| \cdot e^{\mathrm{j}(2\pi n + \arg(z))} \quad \text{for any } n \in \mathbb{Z}$$

Example 4.4.

- $z_1 = 4 + \mathrm{j}\, 4 = 4\sqrt{2} \cdot e^{\mathrm{j}\pi/4}$
- $z_2 = \sqrt{3} - \mathrm{j} = 2 \cdot e^{-\mathrm{j}\pi/6}$
- $z_3 = -1 - \mathrm{j} = \sqrt{2} \cdot e^{-\mathrm{j}3\pi/4} = -\sqrt{2} \cdot e^{\mathrm{j}\pi/4}$

◁

4.4. Basic operations

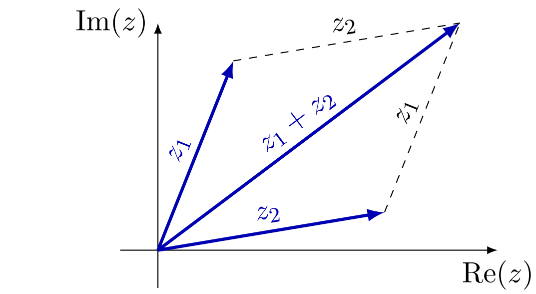
Theorem 4.10 (Basic arithmetic operations)**.** With $z_1 = a_1 + \mathrm{j}\, b_1 \in \mathbb{C}$ and $z_2 = a_2 + \mathrm{j}\, b_2 \in \mathbb{C}$ we have for the basic arithmetic operations:

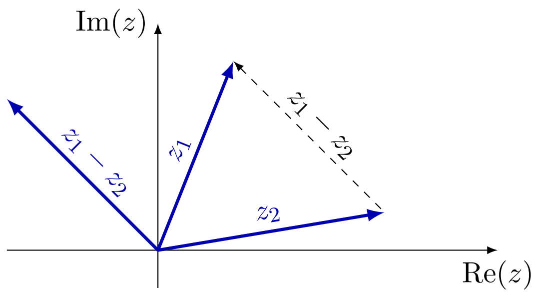
$$
\begin{aligned}
z_1 + z_2 &= a_1 + a_2 + \mathrm{j}(b_1 + b_2) \\
z_1 - z_2 &= a_1 - a_2 + \mathrm{j}(b_1 - b_2) \\
z_1 \cdot z_2 &= a_1 a_2 - b_1 b_2 + \mathrm{j}(a_1 b_2 + a_2 b_1)
\end{aligned}
$$

and for $z_2 \neq 0$:

$$
\begin{aligned}
\frac{z_1}{z_2} &= \frac{a_1 + \mathrm{j}\, b_1}{a_2 + \mathrm{j}\, b_2} \cdot \frac{a_2 - \mathrm{j}\, b_2}{a_2 - \mathrm{j}\, b_2} \\
&= \frac{a_1 a_2 + b_1 b_2}{{a_2}^2 + {b_2}^2} + \mathrm{j} \frac{a_2 b_1 - a_1 b_2}{{a_2}^2 + {b_2}^2}
\end{aligned}
$$

$\lhd$





Example 4.5. For $z_1 = 1 + \mathrm{j}$ and $z_2 = 2 - 2\mathrm{j}$ we have

- $z_1 + z_2 = 3 - \mathrm{j}$
- $z_1 - z_2 = -1 + 3\mathrm{j}$
- $z_1 \cdot z_2 = 4$
- $\frac{z_1}{z_2} = \frac{1}{2}\mathrm{j}$

$\lhd$

Remark: The multiplication and division of complex numbers are difficult to illustrate on the complex plane. They are better understood in polar coordinates where the absolutes are multiplied/divided and the angles are added/subtracted, respectively.

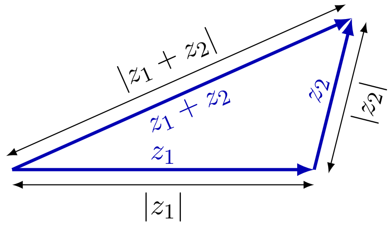
With $z_1 = |z_1| e^{\mathrm{j}\varphi_1}$ and $z_2 = |z_2| e^{\mathrm{j}\varphi_2}$ we have:

$$
\begin{aligned}
z_1 \cdot z_2 &= |z_1|\,|z_2| e^{\mathrm{j}\,(\varphi_1 + \varphi_2)} \\
\frac{z_1}{z_2} &= \frac{|z_1|}{|z_2|} e^{\mathrm{j}\,(\varphi_1 - \varphi_2)}
\end{aligned}
$$

Theorem 4.11 (Properties of absolute)**.** With $z, z_1, z_2 \in \mathbb{C}$ we have

- $|z| \geq 0$
- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$
- $|z_1 + z_2| \leq |z_1| + |z_2|$ (triangle inequality)

$\lhd$

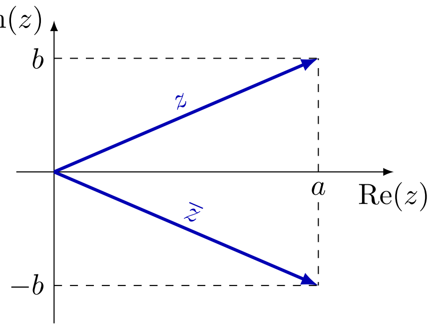


Example 4.6. For $z_1 = 1 + \mathrm{j}$ and $z_2 = 2 - 2\mathrm{j}$ we have

- $|z_1| = \sqrt{2} \geq 0 \quad$ and $\quad |z_2| = \sqrt{8} \geq 0$
- $|z_1 \cdot z_2| = 4 = \sqrt{2} \cdot \sqrt{8} = |z_1| \cdot |z_2|$
- $|z_1 + z_2| = \sqrt{10} \leq \sqrt{2} + \sqrt{8} = |z_1| + |z_2|$

$\lhd$

Definition 4.12 (Conjugation)**.** The *conjugate* of a complex number $z = a + \mathrm{j}\, b \in \mathbb{C}$ is defined by $\overline{z} = a - \mathrm{j}\, b$. A pair z and $\overline{z}$ is called *complex conjugates*. $\lhd$



Theorem 4.13 (Calculating with conjugates)**.** With $z \in \mathbb{C}$ we have

- $\overline{\overline{z}} = z$
- $z \cdot \overline{z} = |z|^2, \quad$ i.e. $z \cdot \overline{z} \geq 0$ and $z \cdot \overline{z} \in \mathbb{R}$
- $z + \overline{z} = 2 \cdot \mathrm{Re}(z)$
- $z - \overline{z} = 2\mathrm{j} \cdot \mathrm{Im}(z)$

◁

Example 4.7. For $z = 2 - 3\mathrm{j}$ we have:

- $\overline{\overline{z}} = 2 - 3\mathrm{j}$
- $z \cdot \overline{z} = 13$
- $z + \overline{z} = 4$
- $z - \overline{z} = -6\mathrm{j}$

◁

Theorem 4.14 (Powers of complex numbers)**.** With $z = |z|\, e^{\mathrm{j}\,\varphi} \in \mathbb{C}$ and $n \in \mathbb{Z}$ we have

$$z^n = |z|^n(\cos n\varphi + \mathrm{j}\,\sin n\varphi) = |z|^n e^{\mathrm{j}n\varphi}$$

◁

Example 4.8. For $z_1 = 1 + \mathrm{j}$ and $z_2 = 2 - 2\mathrm{j}$ we have

$$\begin{aligned} {z_1}^2 &= 2\mathrm{j} & {z_2}^2 &= -8\mathrm{j} \\ {z_1}^3 &= -2 + 2\mathrm{j} & {z_2}^3 &= -16 - 16\mathrm{j} \\ {z_1}^4 &= -4 & {z_2}^4 &= -64 \\ {z_1}^{-2} &= -\frac{1}{2}\mathrm{j} & {z_2}^{-2} &= \frac{1}{8}\mathrm{j} \end{aligned}$$

◁

Definition 4.15 (n^{th}-root of complex numbers)**.** $z \in \mathbb{C}$ is called the n^{th}-root of $y \in \mathbb{C}$ if

$$z^n = y$$

◁

Remark: Care must be taken when working with roots in the context of complex numbers, e.g.

$$1 = \sqrt{1} = \sqrt{(-1) \cdot (-1)} \neq \sqrt{-1} \cdot \sqrt{-1} = \mathrm{j} \cdot \mathrm{j} = -1$$

In fact, the standard root function is defined for non-negative real numbers only: $\sqrt{\cdot} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. When applied to negative real or complex numbers we get more than one solution and, strictly speaking, the root operator is not a function anymore.

The next theorem deals more general with roots for complex numbers.

Theorem 4.16 (Number of roots)**.** For $z \in \mathbb{C}$, $y = |y|e^{\mathrm{j}\,\varphi} \in \mathbb{C}_{\neq 0}$ and $n \in \mathbb{N}$ the equation

$$z^n = y$$

has exactly n solutions for z which are:

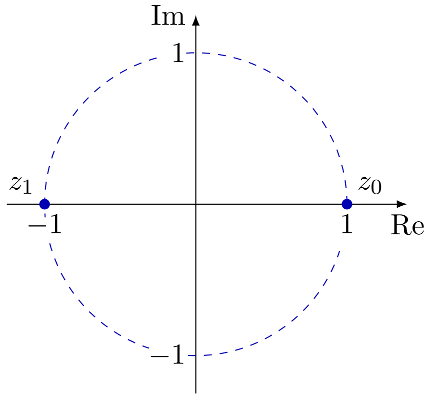
$$z_k = \sqrt[n]{|y|}\exp(\mathrm{j}\tfrac{\varphi + 2\pi k}{n}) \quad \text{for } k = 0, 1, \ldots, n-1$$

◁

Example 4.9. n^{th} root of 1. From real numbers we know already that the square root ($n = 2$), i.e. $z^2 = 1$, has two solutions. Treated as complex number we have $|y| = 1$ and $\varphi = 0$ and get:

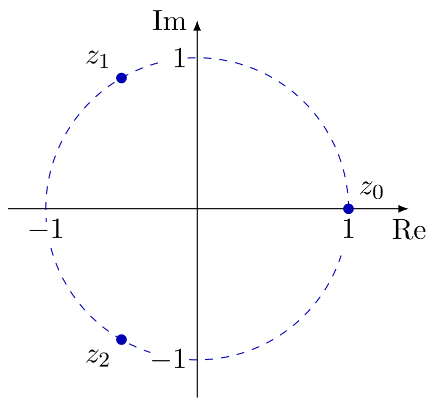
$$\begin{aligned} z_0 &= \sqrt[2]{1}\exp\left(\mathrm{j}\tfrac{0+2\pi\cdot 0}{2}\right) = e^{\mathrm{j}0} = 1 \\ z_1 &= \sqrt[2]{1}\exp\left(\mathrm{j}\tfrac{0+2\pi\cdot 1}{2}\right) = e^{\mathrm{j}\pi} = -1 \end{aligned}$$

In exponential form we have the two angles $\{0, \pi\}$ and the two roots $\{1, -1\}$.



For cubic roots ($n = 3$), i.e. $z^3 = 1$, we have three solutions:

$$\begin{aligned} z_0 &= \sqrt[3]{1}\exp\left(\mathrm{j}\tfrac{0+2\pi\cdot 0}{3}\right) = e^{\mathrm{j}0} = 1 \\ z_1 &= \sqrt[3]{1}\exp\left(\mathrm{j}\tfrac{0+2\pi\cdot 1}{3}\right) = e^{2\mathrm{j}\pi/3} = -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\mathrm{j} \\ z_2 &= \sqrt[3]{1}\exp\left(\mathrm{j}\tfrac{0+2\pi\cdot 2}{3}\right) = e^{4\mathrm{j}\pi/3} = -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}\mathrm{j} \end{aligned}$$



For other $n \in \mathbb{N}$ the n roots are evenly distributed on the unit circle on the complex plane. I.e. for $z^n = 1$ we have

$$z_k = e^{\mathrm{j}2\pi k/n} \quad \text{for } k = 0, 1, \ldots, n-1$$

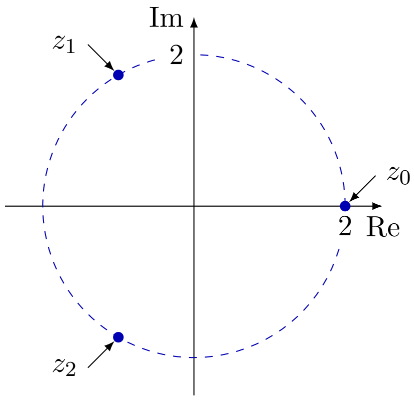
◁

Example 4.10. What are the solutions of

$$z^3 = 8\ ?$$

W.r.t the last theorem we have $|y| = 8$, $\varphi = 0$ and $n = 3$. Hence we get:

$$\begin{aligned} z_k &= \sqrt[3]{8}\,\exp\left(\mathrm{j}\tfrac{2\pi k}{3}\right) = 2e^{\mathrm{j}2\pi k/3} \\ z_0 &= 2e^{\mathrm{j}0} = 2 \\ z_1 &= 2e^{\mathrm{j}\,2\pi/3} = -1 + \sqrt{3}\,\mathrm{j} \\ z_2 &= 2e^{\mathrm{j}\,4\pi/3} = -1 - \sqrt{3}\,\mathrm{j} \end{aligned}$$



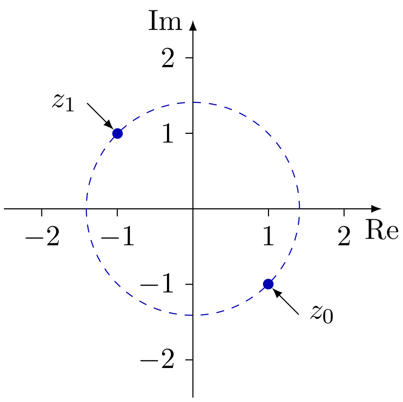
◁

Example 4.11. What are the solutions of

$$z^2 = -2\mathrm{j}\ \ ?$$

W.r.t the last theorem we get $|y| = 2$, $\varphi = -\frac{\pi}{2}$ and $n = 2$. Hence we get:

$$\begin{aligned} z_k &= \sqrt[2]{2}\,\exp\left(\mathrm{j}\frac{-\pi/2+2\pi k}{2}\right) \\ z_0 &= \sqrt{2}\,e^{-\mathrm{j}\,\pi/4} = 1 - \mathrm{j} \\ z_1 &= \sqrt{2}\,e^{\mathrm{j}\,3\pi/4} = -1 + \mathrm{j} \end{aligned}$$



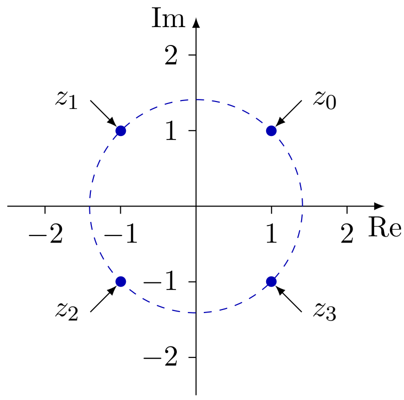
◁

Example 4.12. What are the solutions of

$$z^4 = -4\ \ ?$$

W.r.t the last theorem we have $|y| = 4$, $\varphi = \pi$ and $n = 4$. Hence we get:

$$\begin{aligned} z_k &= \sqrt[4]{4}\,\exp\left(\mathrm{j}\tfrac{\pi+2\pi k}{4}\right) \\ z_0 &= \sqrt{2}\,e^{\mathrm{j}\,\pi/4} = 1 + \mathrm{j} \\ z_1 &= \sqrt{2}\,e^{\mathrm{j}\,3\pi/4} = -1 + \mathrm{j} \\ z_2 &= \sqrt{2}\,e^{\mathrm{j}\,5\pi/4} = -1 - \mathrm{j} \\ z_3 &= \sqrt{2}\,e^{\mathrm{j}\,7\pi/4} = 1 - \mathrm{j} \end{aligned}$$



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4.5. Problems

Problem 4.1: For $n \in \mathbb{Z}$ simplify the following expressions:

1. j^2
2. j^5
3. j^{4n}
4. j^{4n+3}
5. j^{2n}
6. j^{-3}

Problem 4.2: Solve the following equations:

1. $\mathrm{Re}(1 - \mathrm{j})$
2. $\mathrm{Im}(3 + 2\mathrm{j})$
3. $\mathrm{Re}(-a + b\mathrm{j})$
4. $\mathrm{Im}(\mathrm{j}^2)$

Problem 4.3: Plot the following complex numbers in the complex plane:

1. $z_1 = 1 + 2\mathrm{j}$
2. $z_2 = 2 - \mathrm{j}$
3. $z_3 = -3 + 2\mathrm{j}$
4. $z_4 = -2 - 2\mathrm{j}$

Problem 4.4: With $z = a + \mathrm{j}\,b = |z| \cdot e^{\mathrm{j}\varphi}$ complete the following table:

a	b	$\lvert z \rvert$	φ
2	2		
-3	3		
		2	$\pi/2$
		4	$-\pi/4$

Problem 4.5: Plot the following complex numbers in the complex plane:

1. $z_1 = \sqrt{2}\, e^{\mathrm{j}\pi/4}$
2. $z_2 = \sqrt{8}\, e^{-\mathrm{j}3\pi/4}$
3. $z_3 = 2\, e^{-\mathrm{j}\pi/2}$
4. $z_4 = e^{\mathrm{j}\pi}$

Problem 4.6: For $z_1 = 2 + 3\mathrm{j}$ and $z_2 = 3 - 2\mathrm{j}$ solve the following expressions:

1. $z_1 + z_2$
2. $z_2 - z_1$
3. $z_1 \cdot z_2$
4. z_1 / z_2
5. ${z_1}^3$
6. $(z_2/z_1)^2$
7. $|z_1|^2$
8. $|\mathrm{j}z_1 + z_2|$

Problem 4.7: For $z_1 = 2\, e^{\mathrm{j}\pi/4}$ and $z_2 = e^{\mathrm{j}\pi/3}$ solve the following expressions:

1. $z_1 \cdot z_2$
2. ${z_1}^2$
3. ${z_2}^2$
4. ${z_1}^2 \cdot {z_2}^3$
5. $\mathrm{Re}({z_2}^2)$
6. $\mathrm{Im}({z_1}^2)$
7. z_2/z_1
8. $|z_1|$
9. $|{z_2}^3|$
10. $|\mathrm{j}z_1|^2$

Problem 4.8: For $z_1 = 2 + 3\mathrm{j}$ and $z_2 = 3 - 2\mathrm{j}$ solve the following expressions:

1. $z_1 + \overline{z_1}$
2. $\overline{z_2} - z_2$
3. $z_1 - \overline{z_2}$
4. $z_1 \cdot \overline{z_1} + z_2 \cdot \overline{z_2}$
5. $z_1 \cdot \overline{z_1} - |z_1| \cdot |z_2|$
6. $|z_1 \cdot \overline{z_2}|$

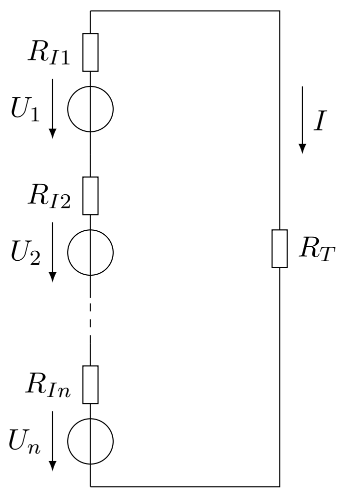
Problem 4.9: Solve the following equations for all possible $z \in \mathbb{C}$:

1. $z^4 = 1$
2. $z^2 = -1$
3. $z^4 = 25$
4. $z^4 = -324$
5. $z^8 = 81$
6. $z^2 = \mathrm{j}$

5. Sequences

5.1. Introduction

Example 5.1. In electronics we deal with resistors and voltage supplies. Let's assume a terminating resistor R_T and a variable number of voltage supplies with equal voltage $U_1 = U_2 = \ldots = U$ and equal internal resistance $R_{I1} = R_{I2} = \ldots = R_I$. The voltage supplies are connected in a chain, i.e. the voltage and internal resistance increases with increasing number of voltage supplies.



What is the total current I through the terminating resistor R_T? For one voltage supply we get by Ohm's law:

$$I_1 = \frac{U}{R_I + R_T}$$

For two voltage supplies we get:

$$I_2 = \frac{U + U}{R_I + R_I + R_T} = \frac{2 \cdot U}{2 \cdot R_I + R_T}$$

For an arbitrary number n of voltage supplies we get:

$$I_n = \frac{n \cdot U}{n \cdot R_I + R_T} = \frac{U}{R_I + \frac{R_T}{n}}$$

To find the maximum possible electrical current we increase the number of voltage supplies towards infinity:

$$I_{\max} = \lim_{n \to \infty} \frac{U}{R_I + \frac{R_T}{n}} = \frac{U}{R_I}$$

Hence, for every number of voltage supplies we found an associated total current and a limiting value for an infinite number of voltage supplies.

Let's assume the voltage supplies have a voltage of 1 V with an internal resistor of 1 Ω. With a terminating resistor of 3 Ω we get for I in ampere:

$$I_1 = \tfrac{1}{4} \qquad I_2 = \tfrac{2}{5} \qquad I_3 = \tfrac{3}{6} \qquad I_4 = \tfrac{4}{7} \qquad \ldots$$

Noted as a sequence we write in short:

$$(I_n) = (\tfrac{1}{4}, \tfrac{2}{5}, \tfrac{3}{6}, \tfrac{4}{7}, \tfrac{5}{8}, \tfrac{6}{9}, \tfrac{7}{10}, \tfrac{8}{11}, \ldots)$$

◁

5.2. Definition of sequences

Remark: Most of the concepts discussed here are valid for real and complex numbers. Therefore we will use $\mathbb{K}$ which stands for either $\mathbb{R}$ or $\mathbb{C}$. Every statement using $\mathbb{K}$ is valid for $\mathbb{R}$ and $\mathbb{C}$.

Definition 5.1 (Sequence)**.** A *sequence* maps $\mathbb{N}$ to $\mathbb{K}$. We write $(x_n)_{n \in \mathbb{N}}$ or just (x_n) where every $n \in \mathbb{N}$ is mapped to a real or complex number x_n. ◁

Example 5.2. Some sequences:

- The sequence (a_n) with $a_n = \frac{1}{n!}$ is a sequence of real numbers:

$$a_1 = \tfrac{1}{1!},\ a_2 = \tfrac{1}{2!},\ a_3 = \tfrac{1}{3!},\ a_4 = \tfrac{1}{4!},\ \ldots$$
$$(a_n) = (\tfrac{1}{1!}, \tfrac{1}{2!}, \tfrac{1}{3!}, \tfrac{1}{4!}, \ldots)$$

- The sequence (b_n) with $b_n = 2$ is an example of a *constant sequence*:

$$(b_n) = (2, 2, 2, 2, 2, \ldots)$$

- The sequence (c_n) with $c_n = \mathrm{j}^n$ is complex and periodic:

$$(c_n) = (\mathrm{j}, -1, -\mathrm{j}, 1, \mathrm{j}, -1, -\mathrm{j}, 1, \ldots)$$

◁

Definition 5.2 (Bounded sequence)**.** A sequence (a_n) is said to be *bounded* if there exists a limit $M \in \mathbb{R}$ such that

$$|a_n| \leq M \qquad \text{for all } n \in \mathbb{N}$$

◁

Example 5.3.

- (a_n) with $a_n = \frac{1}{n}$ is bounded
- (b_n) with $b_n = n^2$ is not bounded
- (c_n) with $c_n = (-1)^n$ is bounded
- (d_n) with $d_n = x^n$, $x \in \mathbb{C}$ is bounded for $|x| \leq 1$

◁

Definition 5.3 (Monotonicity)**.** Let (a_n) be a sequence of real numbers. (a_n) is called

$$\begin{array}{rcl} \textit{increasing} & \text{if} & a_{n+1} \geq a_n \\ \textit{decreasing} & \text{if} & a_{n+1} \leq a_n \\ \textit{strictly increasing} & \text{if} & a_{n+1} > a_n \\ \textit{strictly decreasing} & \text{if} & a_{n+1} < a_n \end{array}$$

for all $n \in \mathbb{N}$. If a sequence is (strictly) increasing or (strictly) decreasing we call it (strictly) *monotone*. ◁

Remark: This definition on monotonicity is not applicable for complex sequences since complex numbers can not be ordered.

Example 5.4.

- (a_n) with $a_n = \frac{1}{n}$ is strictly decreasing
- (b_n) with $b_n = n^2$ is strictly increasing
- (c_n) with $c_n = (-1)^n$ is not monotone
- (d_n) with $d_n = x^n$ is monotone for $x \in \mathbb{R}_{\geq 0}$ and strictly monotone for $x \in \mathbb{R}_{>0} \setminus \{1\}$

◁

5.3. Convergence and divergence

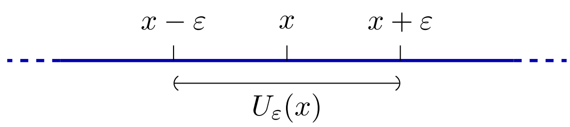
Definition 5.4 (Epsilon-neighbourhood)**.** For a number $x \in \mathbb{K}$ and $\varepsilon \in \mathbb{R}_{>0}$ we say

$$U_\varepsilon(x) = \{y \in \mathbb{K} \mid |y - x| < \varepsilon\}$$

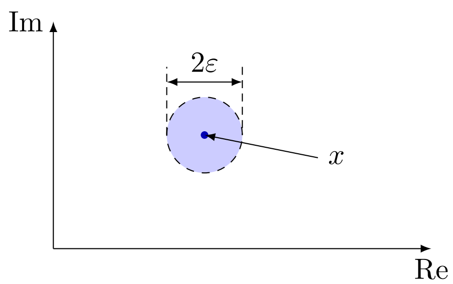
is the *epsilon-neighbourhood* of x. ◁

Remark: For real numbers the epsilon-neighbourhood equals the open interval between $x - \varepsilon$ and $x + \varepsilon$:

$$U_\varepsilon(x) = (x - \varepsilon, x + \varepsilon) \quad \text{for } x \in \mathbb{R}.$$



For complex numbers the epsilon-neighbourhood equals a circular disc on the complex plane with diameter 2ε. All numbers inside this disc (excluding the edge) belong to the epsilon-neighbourhood of x.



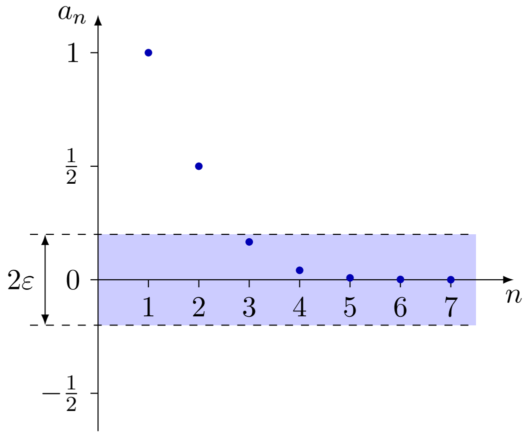
Definition 5.5 (Convergence)**.** A sequence (a_n) with $a_n \in \mathbb{K}$ is said to be *convergent* with limiting value a if for any $\varepsilon \in \mathbb{R}_{>0}$ there exist an $N \in \mathbb{N}$ such that all a_n with $n \geq N$ are elements of the epsilon-neighbourhood $U_\varepsilon(a)$. In short:

$$\forall \varepsilon \in \mathbb{R}_{>0}\ \exists N \in \mathbb{N}\ \forall n \geq N : |a_n - a| < \varepsilon$$

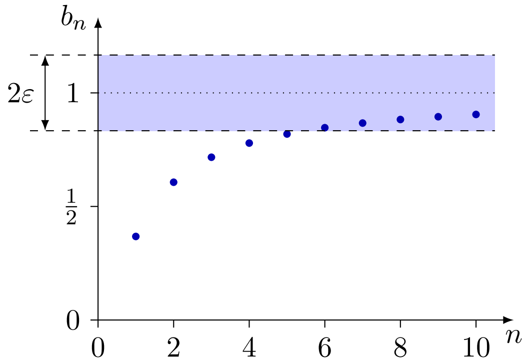
◁

Example 5.5.

- The sequence (a_n) with $a_n = \frac{1}{n!}$ is convergent.



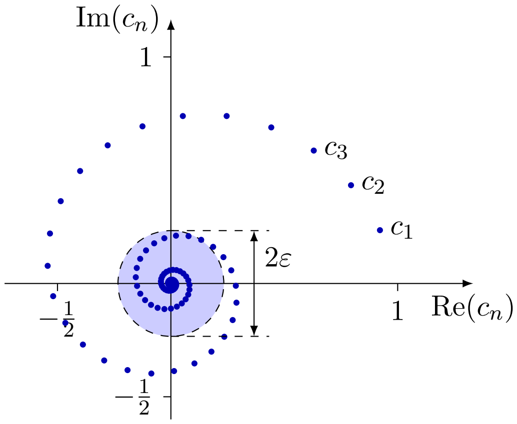
- The sequence (b_n) with $b_n = e^{-1/n}$ is convergent.



- The sequence (c_n) with

$$\begin{aligned} c_n &= e^{-n/20+\mathrm{j}\,n/4} \\ &= e^{-n/20}[\cos(\tfrac{n}{4}) + \mathrm{j}\,\sin(\tfrac{n}{4})] \end{aligned}$$

  is convergent.



◁

Remark: A sequence is said to be a *zero sequence* if its elements converge towards zero. E.g. in the previous example (a_n) and (c_n) are zero sequences.

Definition 5.6 (Divergence)**.** Non-convergent sequences are called *divergent* sequences. ◁

Example 5.6.

- (a_n) with $a_n = n$ is not convergent, i.e. divergent.
- (b_n) with $b_n = \mathrm{j}^n$ is divergent.
- (c_n) with $c_n = x^n$, $x \in \mathbb{R}$ is convergent for $x \in (-1, 1]$ and divergent otherwise.
- (d_n) with $d_n = z^n$, $z \in \mathbb{C}$ is convergent for $|z| < 1 \wedge z = 1$ and divergent otherwise.

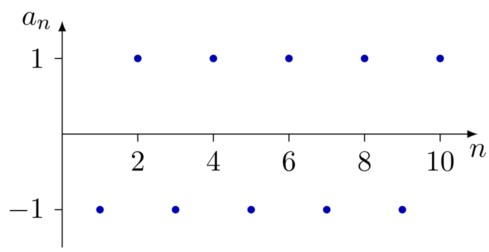
◁

Theorem 5.7 (Convergent sequences bounded)**.** Convergent sequences are bounded. ◁

Remark: This theorem states an *implication*: *If* a sequence is convergent *then* it is bounded too.

The reverse is not valid: A bounded sequence may or may not be convergent.

Example: The sequence (a_n) with $a_n = (-1)^n$ is bounded but not convergent.



Theorem 5.8 (Monotone bounded sequences)**.** Monotone bounded sequences are convergent. ◁

5.4. Limit of sequences

Definition 5.9 (Limit of a sequence)**.** For a convergent sequence (a_n) where the elements a_n converge towards a we write:

$$\lim_{n\to\infty} a_n = a$$

◁

Remark: Other (abbreviated) notations are:

$$\lim a_n = a$$
$$a_n \underset{n\to\infty}{\longrightarrow} a$$
$$a_n \longrightarrow a$$

Example 5.7.

- for (a_n) with $a_n = \frac{1}{n}$ we have
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$
- for (b_n) with $b_n = \frac{n}{n+1}$ we have
$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$
- for (c_n) with $c_n = \frac{n+1}{n}$ we have
$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1} = 1$$
- for (d_n) with $d_n = \frac{x^n}{n!}$, $x \in \mathbb{R}$ we have
$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \frac{x^n}{n!} = 0$$
since for $n > |x|$ the denominator increases faster than the numerator, i.e.
$$|d_{n+1}| < |d_n| \quad \text{for} \quad n > |x|$$
- for (e_n) with $e_n = e^{1/n}$ we have
$$\lim_{n\to\infty} e_n = \lim_{n\to\infty} e^{1/n} = e^0 = 1$$

◁

Remark: Obviously, for a zero sequence (a_n) we have:
$$\lim_{n\to\infty} a_n = 0$$
Hence, (a_n) and (d_n) in the previous example are zero sequences.

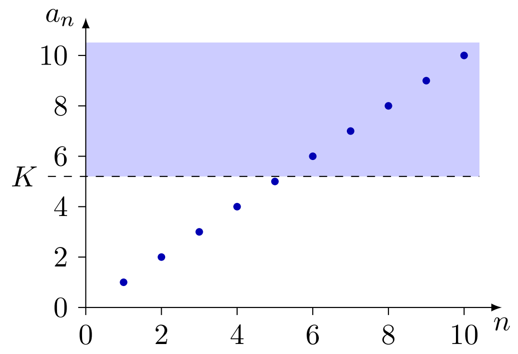
Definition 5.10 (Definite divergence)**.** A sequence (a_n) with $a_n \in \mathbb{R}$ is divergent towards $+\infty$ if for any $K \in \mathbb{R}$ there exist an $N \in \mathbb{N}$ such that for all a_n with $n \geq N$ we have $a_n > K$. In short:
$$\forall K \in \mathbb{R}\; \exists N \in \mathbb{N}\; \forall n \geq N : a_n > K$$

A sequence (a_n) with $a_n \in \mathbb{R}$ is divergent towards $-\infty$ if for any $K \in \mathbb{R}$ there exist an $N \in \mathbb{N}$ such that for all a_n with $n \geq N$ we have $a_n < K$. In short:
$$\forall K \in \mathbb{R}\; \exists N \in \mathbb{N}\; \forall n \geq N : a_n < K$$

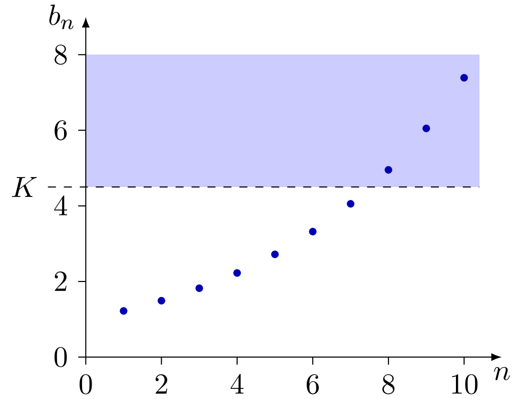
◁

Example 5.8.

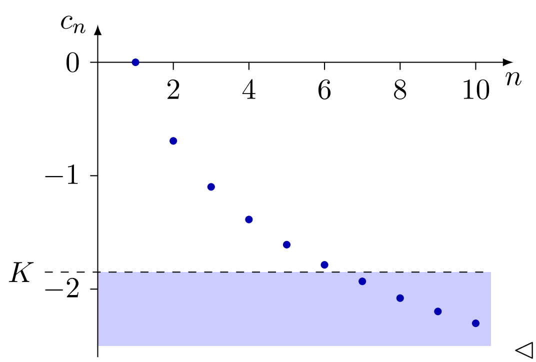
- For (a_n) with $a_n = n$ we have
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n = \infty$$



- For (b_n) with $b_n = \exp\left(\frac{n}{5}\right)$ we have
$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \exp\left(\frac{n}{5}\right) = \infty$$



- For (c_n) with $c_n = \ln(1/n)$ we have
$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \ln\left(\frac{1}{n}\right) = -\infty$$



◁

Theorem 5.11 (Calculating with convergent sequences)**.** Let (a_n) and (b_n) be convergent sequences. We have for the sequence of sums $(a_n + b_n)$ and the sequence of products $(a_n \cdot b_n)$:

$$\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty}(a_n) \pm \lim_{n\to\infty}(b_n)$$
$$\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty}(a_n) \cdot \lim_{n\to\infty}(b_n)$$

If $b_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}(b_n) \neq 0$ then we have for the sequence of quotients $(\frac{a_n}{b_n})$:

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim\limits_{n\to\infty}(a_n)}{\lim\limits_{n\to\infty}(b_n)}$$

◁

Example 5.9. Let (a_n) be a sequence with $a_n = \frac{2n-1}{n}$ and (b_n) be a sequence with $b_n = \cos(\frac{1}{n})$. Then we have:

- $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\frac{2n-1}{n} = \lim\limits_{n\to\infty}\frac{2-\frac{1}{n}}{1} = 2$
- $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty}\cos\left(\frac{1}{n}\right) = 1$
- $\lim\limits_{n\to\infty}(a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n = 3$
- $\lim\limits_{n\to\infty}(a_n - b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n = 1$
- $\lim\limits_{n\to\infty}(a_n \cdot b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n = 2$
- $\lim\limits_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} = 2$

◁

5.5. Problems

Problem 5.1: Evaluate the first five elements of the following sequences:

- (a_n) with $a_n = 3n$
- (b_n) with $b_n = \frac{n+1}{2n-1}$
- (c_n) with $c_n = \frac{n^2+n}{2}$
- (d_n) with $d_n = \sum_{k=1}^{n} k$
- (e_n) with $e_n = \mathrm{j}^{2n}$
- (f_n) with $f_n = \mathrm{j}^{-1}$
- (g_n) with $g_n = 5$

Problem 5.2: Which of the sequences in the previous problem are bounded?

Problem 5.3: Fill the following table with $\surd$ for the appropriate properties of the sequences in problem 5.1.

sequence	increasing	strictly increasing	decreasing	strictly decreasing	monotone	strictly monotone
(a_n)						
(b_n)						
(c_n)						
(d_n)						
(e_n)						
(f_n)						
(g_n)						

Problem 5.4: Describe $U_\varepsilon(2)$ with $\varepsilon = \frac{1}{2}$ for real numbers and plot it on the real line.

Problem 5.5: Describe $U_\varepsilon(\mathrm{j})$ with $\varepsilon = \frac{1}{2}$ for complex numbers and plot it on the complex plane.

Problem 5.6: Describe *convergence* in your own words.

Problem 5.7: Which of the sequences in problem 5.1 are convergent, which of them are divergent?

Problem 5.8: Find the limit of the following sequences:

- (a_n) with $a_n = \frac{1}{n^2} + 1$
- (b_n) with $b_n = \frac{n+1}{2n-1}$
- (c_n) with $c_n = \frac{n^2-1}{2n^2}$
- (d_n) with $d_n = \frac{n^2}{1-2n}$
- (e_n) with $e_n = \tan\left(\frac{\pi n}{2n+1}\right)$

Problem 5.9: For the sequences in the previous problem find the limits of the following expressions:

a) $(a_n + b_n)$ b) $(a_n \cdot c_n)$

c) $(c_n \cdot d_n)$ d) $\left(\frac{e_n}{b_n}\right)$

6. Series

6.1. Introduction

Example 6.1. Achilles and the tortoise.

For some reasons Achilles, the famous runner of the Greek mythology, has a race with a tortoise. Obviously the tortoise has not the slightest chance and, hence, is given a head start of 100 m. Let us assume that Achilles runs (only) twice as fast as the tortoise and both run with a constant speed.

Some time after the start Achilles reaches the 100 m point where the tortoise started. However, the tortoise is not there but ran already 50 m and is at 150 m. So, Achilles runs another 50 m, but again does not reach the tortoise who ran another 25 m. Will Achilles ever reach the tortoise? And if yes, where will it be?

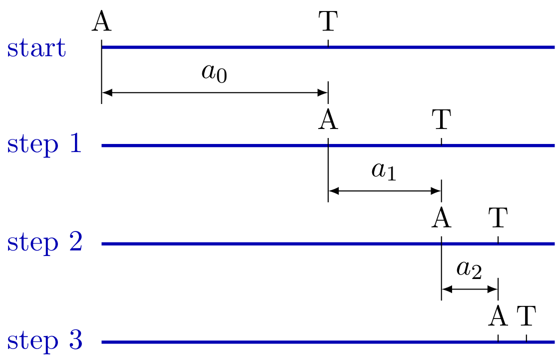
Of course, we could set up an equation and solve it for the point where the two runners meet. However, let us look at it as a sequence of distance steps of Achilles in meter: $a_0 = 100$, $a_1 = 50$, $a_2 = 25$, $a_3 = 12.5$ etc. So, we define a sequence by

$$(a_n) = (100, 50, 25, 12\tfrac{1}{2}, 6\tfrac{1}{4}, \ldots)$$

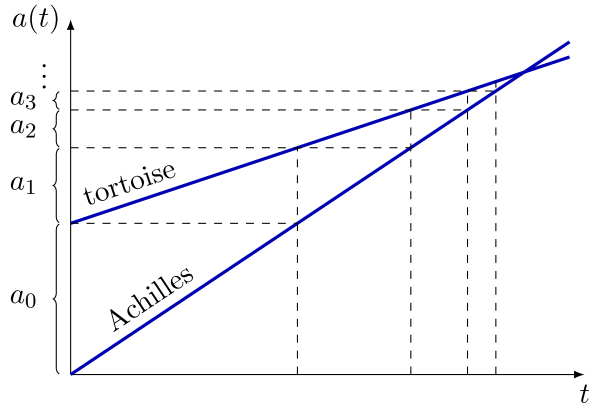
where

$$a_n = 100\left(\frac{1}{2}\right)^n \qquad \text{for } n = 0, 1, 2, \ldots$$

The following sketch shows the first three distance steps of Achilles:



We also may look at it on a diagram where we plot the time on the abscissa an the distance of Achilles and the tortoise on the ordinate:

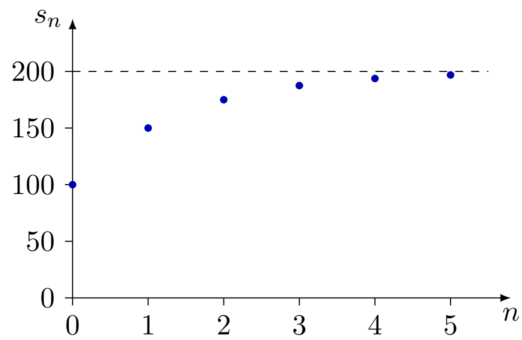


To find the point where Achilles and the tortoise meet we sum up the distance steps which gives us a sequence of partial sums. With

$$s_n = \sum_{k=0}^{n} a_k = 100 \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k$$

we have

$$(s_n) = (100, 150, 175, 187\tfrac{1}{2}, 193\tfrac{3}{4}, \ldots)$$



From theorem 2.21 on page 17 we know

$$\sum_{k=0}^{n-1} q^k = \frac{1-q^n}{1-q}$$

hence with $q = \frac{1}{2}$ we get for the partial sums:

$$s_n = 100 \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = 200\left(1-\left(\tfrac{1}{2}\right)^{n+1}\right)$$

To get the point, where Achilles and the tortoise meet we have to find s_n for an infinite n:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} 200\left(1-\underbrace{\left(\tfrac{1}{2}\right)^{n+1}}_{\to 0}\right) = 200$$

At last, they will meet at the 200m point! ◁

6.2. Definition of series

Definition 6.1 (Partial sum)**.** Let (a_n) be a sequence. The n^{th} *partial sum* s_n is defined by:

$$s_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + \ldots + a_n$$

◁

Remark: If we combine all partial sums to a sequence (s_n) we call it the *sequence of partial sums.*

In the previous chapter we defined that a sequence maps $\mathbb{N}$ to $\mathbb{K}$ (which is $\mathbb{R}$ or $\mathbb{C}$), i.e. the elements of a sequence (a_n) with the subscripts $1, 2, 3, \ldots$. In this chapter we untighten this definition and allow the sequence to start at other values than one, e.g. at zero or two.

Example 6.2.

- For the Gaussian sum we have the sequence of natural numbers

  $$(a_n) = (1, 2, 3, 4, 5, 6, 7, \ldots)$$

  and the partial sums

  $$s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

  which gives us the sequence of partial sums:

  $$(s_n) = (1, 3, 6, 10, 15, 21, 28, \ldots)$$

- For the square numbers we have the sequence of odd numbers

  $$(a_n) = (1, 3, 5, 7, 9, 11, \ldots)$$

  and the partial sums

  $$s_n = \sum_{k=1}^{n} (2k-1) = n^2$$

  which gives us the sequence of partial sums, i.e. in this case the sequence of square numbers:

  $$(s_n) = (1, 4, 9, 16, 25, 36, 49, \ldots)$$

- For the sequence

  $$(a_n) = \left(\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \tfrac{1}{32}, \ldots\right)$$

  we get the partial sums

  $$s_n = \sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

  which gives us the sequence of partial sums:

  $$(s_n) = \left(\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{7}{8}, \tfrac{15}{16}, \tfrac{31}{32}, \ldots\right)$$

◁

Definition 6.2 (Series)**.** Let (a_n) be a sequence. We call

$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + a_3 + \ldots$$

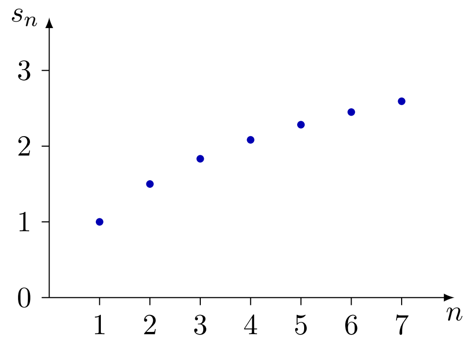
an (infinite) *series.* ◁

Remark: We call a_n the n^{th} *series element*. If a sequence (a_n) starts with an index greater than zero the first few series elements are treated as zero.

Example 6.3.

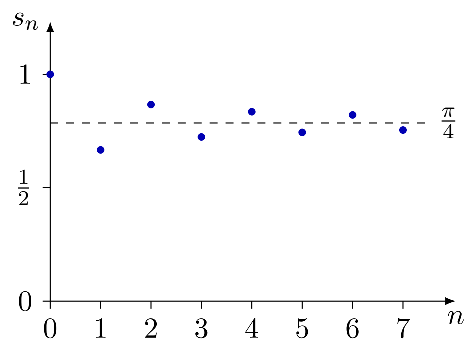
- Harmonic series:

  $$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$



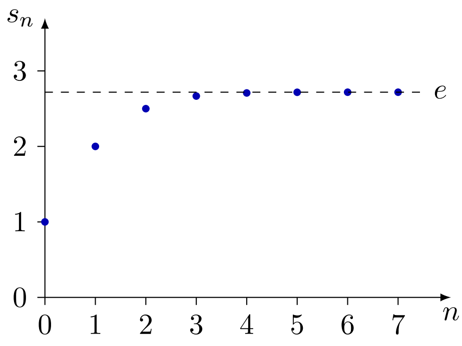
- Leibniz series:

  $$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$



- Euler's number:

$$\sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$
$$= 2.71828\dots$$



◁

6.3. Convergence of series

Definition 6.3 (Convergence and divergence)**.** Let (a_n) be a sequence and (s_n) the corresponding sequence of partial sums, i.e. $s_n = \sum_{k=0}^{n} a_k$. If the sequence (s_n) is *convergent* we say the series $\sum_{k=0}^{\infty} a_k$ is convergent and the limit is given by:

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} s_n$$

Series that are not convergent are called *divergent.* ◁

Example 6.4.

- The series $\sum_{k=0}^{\infty} a_k$ with $a_k = \frac{1}{k!}$ is convergent with Euler's number as limiting value:

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{1}{k!} = e = 2.71828\dots$$

- The *harmonic series* $\sum_{k=1}^{\infty} b_k$ with $b_k = \frac{1}{k}$ is divergent:

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

This becomes clear when looking at the partial sums s_{2^n}, $n = 0, 1, 2, \dots$, i.e. s_1, s_2, s_4, s_8, s_{16} etc. The difference between these partial sums is $\geq \frac{1}{2}$.

$$1 + \underbrace{\tfrac{1}{2}}_{\geq \frac{1}{2}} + \underbrace{\tfrac{1}{3} + \tfrac{1}{4}}_{\geq \frac{1}{2}} + \underbrace{\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}}_{\geq \frac{1}{2}} + \dots$$

Hence we have

$$s_{2^n} \geq 1 + \tfrac{n}{2} \qquad \text{for } n = 1, 2, 3, \dots$$

Since the this term has a limit of ∞ the harmonic series diverges towards ∞ too.

- The series $\sum_{k=1}^{\infty} c_k$ with $c_k = \frac{(-1)^k}{k}$ is convergent:

$$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} = -\ln(2)$$

◁

Theorem 6.4 (Calculating with series)**.** Let $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ be convergent series and $c \in \mathbb{K}$. Then the series $\sum_{k=0}^{\infty} (a_k + b_k)$ and $\sum_{k=0}^{\infty} c\, a_k$ are convergent too and we have:

$$\sum_{k=0}^{\infty} (a_k + b_k) = \sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k$$
$$\sum_{k=0}^{\infty} c\, a_k = c \sum_{k=0}^{\infty} a_k$$

◁

Example 6.5. With the two sequences (a_n), $a_n = \frac{1}{2^n}$ and (b_n), $b_n = (-1)^n \frac{1}{2^n}$ both for $n = 0, 1, 2, \dots$ and $c = 3 + \mathrm{j}$ we have:

$$\sum_{k=0}^{\infty} a_k = 2 \qquad \text{and} \qquad \sum_{k=0}^{\infty} b_k = \tfrac{2}{3}$$

Applying the previous theorem we get:

- $\displaystyle\sum_{k=0}^{\infty} (a_k + b_k) = \sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k = 2\tfrac{2}{3}$
- $\displaystyle\sum_{k=0}^{\infty} c\, a_k = c \sum_{k=0}^{\infty} a_k = 6 + 2\mathrm{j}$
- $\displaystyle\sum_{k=0}^{\infty} c\, b_k = c \sum_{k=0}^{\infty} b_k = 2 + \tfrac{2}{3}\mathrm{j}$

◁

Example 6.6. Let (a_n) be a sequence with

$$a_n = \frac{1}{2^{n+1} + 2^n \mathrm{j}}$$

What is the limiting value for the sum of all elements of (a_n)?

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{1}{2^{k+1} + 2^k \mathrm{j}} = \sum_{k=0}^{\infty} \frac{1}{2 \cdot 2^k + 2^k \mathrm{j}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(2+\mathrm{j}) \cdot 2^k} = \frac{1}{2+\mathrm{j}} \sum_{k=0}^{\infty} \frac{1}{2^k}$$
$$= \frac{1}{2+\mathrm{j}} \cdot 2 = \frac{2}{2+\mathrm{j}} \cdot \frac{2-\mathrm{j}}{2-\mathrm{j}} = \frac{4-2\mathrm{j}}{4+1}$$
$$= \tfrac{4}{5} - \tfrac{2}{5}\mathrm{j}$$

◁

6.4. Geometric series

Theorem 6.5 (Geometric sum). We call

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + x^3 + \ldots + x^n$$

geometric sum. For any $x \in \mathbb{K}_{\neq 1}$ we have:

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}$$

◁

Remark: The term *geometric sum* comes from the fact that each summand is the *geometric mean* of the two neighbour summands:

$$a_k = \sqrt{a_{k-1} a_{k+1}}$$

i.e.

$$a_k = \sqrt{x^{k-1} x^{k+1}} = \sqrt{x^{2k}} = x^k$$

Example 6.7.

- $\displaystyle\sum_{k=0}^{7} 3^k = \frac{1-3^8}{1-3} = \frac{-6560}{-2} = 3280$
- $\displaystyle\sum_{k=0}^{4} \frac{1}{2^k} = \sum_{k=0}^{4} \left(\frac{1}{2}\right)^k = \frac{1-\left(\frac{1}{2}\right)^5}{1-\frac{1}{2}} = 1\tfrac{15}{16}$
- $\displaystyle\sum_{k=0}^{3} \frac{1}{10^k} = \frac{1-\left(\frac{1}{10}\right)^4}{1-\frac{1}{10}} = \tfrac{1111}{1000} = 1.111$

◁

Example 6.8. In computer science unsigned binary integers consist of a number of binary digits each with the two possible values zero and one. The rightmost digit (bit 0) is weighted with a factor 1 (i.e. 2^0), the second digit from the right (bit 1) has the factor 2 (i.e. 2^1), the third (bit 2) the factor 4 (i.e. 2^2) and so on.

What is the greatest number to be stored in unsigned integers with n bits? We notice that we deal with the geometric sum and that the leftmost bit has the number $n-1$.

$$\sum_{k=0}^{n-1} 2^k = \frac{1-2^n}{1-2} = 2^n - 1$$

Some typical examples:

- 1 byte = 8 bit: 255
- 2 byte = 16 bit: 65 535
- 4 byte = 32 bit: 4 294 967 295

◁

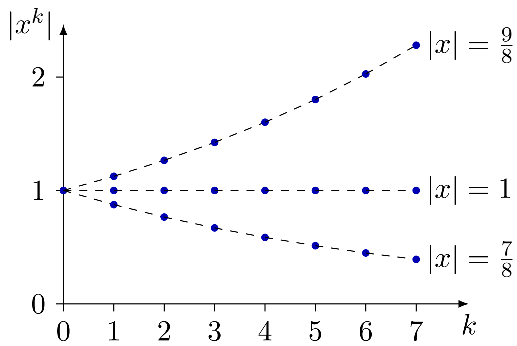
Theorem 6.6 (Geometric series). The *geometric series*

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots$$

with $x \in \mathbb{K}$ is convergent for $|x| < 1$ and we have:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

◁



Example 6.9. We come back to example 6.1 in the introduction of this chapter. There we found the sequence (a_n) with $a_n = 100\left(\frac{1}{2}\right)^n$ for $n = 0, 1, 2, \ldots$. To find the meeting point of Achilles and the tortoise we take the infinite geometric sum:

$$\sum_{k=0}^{\infty} a_k = 100 \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{100}{1-\frac{1}{2}} = 200$$

◁

Example 6.10. What is 0.999 999 ...? Each digit behind the dot has a weighting factor of $\frac{1}{10}$ of its left neighbour. Hence, we may write:

$$\sum_{k=1}^{\infty} \frac{9}{10^k} = \sum_{k=0}^{\infty} \frac{9}{10^k} - 9 = 9 \sum_{k=0}^{\infty} \left(\frac{1}{10}\right)^k - 9$$

$$= 9\frac{1}{1-\frac{1}{10}} - 9 = 1$$

◁

Remark: The fact that the geometric series is convergent for $|x| < 1$ serves as a tool to prove convergence of arbitrary series, see theorem 6.11 and 6.12. There we use two properties of the convergent geometric series:

$$\sqrt[n]{|a_n|} = \sqrt[n]{|x^n|} = |x| < 1$$
$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{x^n}\right| = |x| < 1$$

6.5. Convergence criteria

Theorem 6.7 (Convergent series)**.** If a series $\sum_{k=0}^{\infty} a_k$ is convergent, then the sequence (a_n) is a zero sequence. ◁

Remark: This is an implication and, hence, is not valid in the other direction. However, due to this theorem a necessary but not sufficient constraint for convergence of a series is that the corresponding sequence (a_n) is a zero sequence.

Example 6.11.

- For all examples with convergent series $\sum_{k=0}^{\infty} a_k$ in this chapter the sequences (a_n) are zero sequences.
- The *harmonic series* $\sum_{k=1}^{\infty} a_k$ with $a_k = \frac{1}{k}$ is divergent, however, the sequence (a_n) is a zero sequence.

◁

Definition 6.8 (Absolute convergence)**.** Let $\sum_{k=0}^{\infty} a_k$ be a series. If the series $\sum_{k=0}^{\infty} |a_k|$ is convergent, then we call $\sum_{k=0}^{\infty} a_k$ *absolute convergent*. ◁

Theorem 6.9 (Absolute convergent series are convergent)**.** Every absolute convergent series is convergent. ◁

Remark: If we find that a series is convergent for the absolute of its elements it is convergent for any signs of the elements. However, if a series is absolute divergent it may or may not be convergent.

Example 6.12.

- The series $\sum_{k=1}^{\infty} a_k$ with $a_k = \pm\frac{1}{k^2}$ is absolute convergent and, hence, is convergent for any signs of a_k.
- The series $\sum_{k=1}^{\infty} a_k$ with $a_k = \frac{(-1)^{k+1}}{k}$ is absolute divergent (harmonic series) but still convergent with the limit $\ln 2$.

◁

Theorem 6.10 (Comparison test)**.** If the series $\sum_{k=0}^{\infty} b_k$ is absolute convergent, then the series $\sum_{k=0}^{\infty} a_k$ is absolute convergent too if for almost all summands we have $|a_k| \leq |b_k|$.

Let $\sum_{k=0}^{\infty} b_k$ be an absolute divergent series. The series $\sum_{k=0}^{\infty} a_k$ is absolute divergent too if for almost all summands we have $|a_k| \geq |b_k|$. (Although being absolute divergent it still may be convergent for appropriate signs of the summands.) ◁

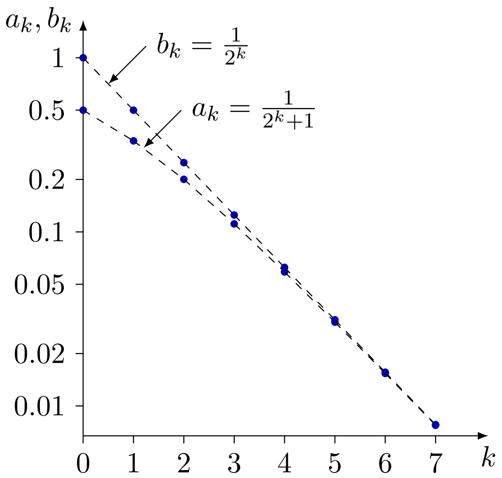
Remark: The term *almost all* in the context of an infinite number of elements means "all, except a finite number of elements".

Example 6.13. Is the series $\sum_{k=0}^{\infty} a_k$ with $a_k = \frac{1}{2^k+1}$ convergent? We look for a series that serves as a limit for the comparison test.

The geometric series $\sum_{k=0}^{\infty} b_k$ with $b_k = \frac{1}{2^k}$ is convergent and we have:

$$|a_k| \leq |b_k| \qquad \text{for all } k \in \mathbb{N}$$

Hence, the series $\sum_{k=0}^{\infty} a_k$ is convergent.



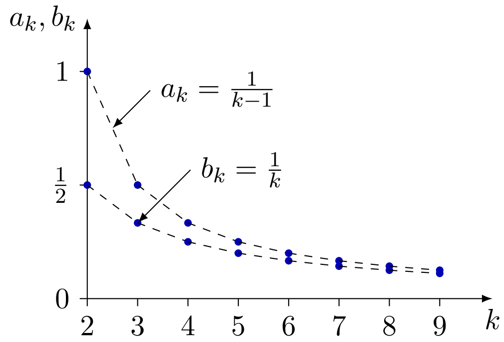
◁

Example 6.14. Is the series $\sum_{k=2}^{\infty} a_k$ with $a_k = \frac{1}{k-1}$ convergent? We look for a series that serves as a limit for the comparison test.

The harmonic series $\sum_{k=1}^{\infty} b_k$ with $b_k = \frac{1}{k}$ is divergent and we have:

$$|a_k| \geq |b_k| \qquad \text{for all } k \in \mathbb{N}, k \geq 2$$

Hence, the series $\sum_{k=2}^{\infty} a_k$ is absolute divergent. Since all a_k are positive the series is divergent.



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Theorem 6.11 (Root test)**.** Let $\sum_{k=0}^{n} a_k$ be a series. If there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\sqrt[n]{|a_n|} < 1$$

the series is absolute convergent. If there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\sqrt[n]{|a_n|} > 1$$

the series is divergent. I.e. for

$$q = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

if $q < 1$ the series is absolute convergent, if $q > 1$ the series is divergent and if $q = 1$ the root test does not give evidence on convergence. ◁

Remark: The root test has been developed by Augustin-Louis Cauchy and, hence, is sometimes called *Cauchy root test* or *Cauchy's radical test*.

Example 6.15. Is the series $\sum_{k=0}^{\infty} a_k$ with

$$a_k = \frac{\sqrt{k}}{2^k}$$

convergent? We apply the root test:

$$q = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{\sqrt{n}}{2^n}\right|}$$

$$= \lim_{n\to\infty} \sqrt[n]{\sqrt{n} \cdot \left(\tfrac{1}{2}\right)^n} = \lim_{n\to\infty} \sqrt[n]{\sqrt{n}} \cdot \sqrt[n]{\left(\tfrac{1}{2}\right)^n}$$

$$= \lim_{n\to\infty} \sqrt{\sqrt[n]{n}} \cdot \tfrac{1}{2} = \tfrac{1}{2}\sqrt{\lim_{n\to\infty} \sqrt[n]{n}}$$

Now we focus on the term $\sqrt[n]{n}$

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} \exp\left(\ln(n^{\frac{1}{n}})\right)$$

$$= \lim_{n\to\infty} \exp\left(\frac{\ln(n)}{n}\right)$$

$$= \lim_{n\to\infty} \exp\left(\frac{\ln(n)}{\exp(\ln(n))}\right)$$

With $y = \ln(n)$ and $\lim_{n\to\infty} y = \infty$ we get the limit for the large brackets:

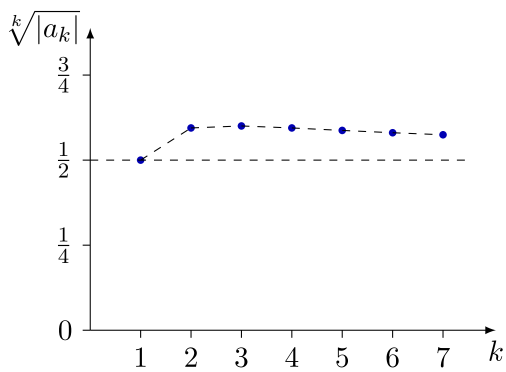
$$\lim_{y\to\infty} \frac{y}{e^y} = \lim_{y\to\infty} \frac{y}{1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \cdots}$$

$$= \lim_{y\to\infty} \frac{1}{\frac{1}{y} + 1 + \frac{y}{2} + \frac{y^2}{6} + \cdots} = 0$$

Hence, we get for q:

$$q = \tfrac{1}{2}\sqrt{\lim_{n\to\infty} \sqrt[n]{n}} = \tfrac{1}{2}\sqrt{e^0} = \frac{1}{2}$$

Therefore, the series is convergent with q having the limit $\frac{1}{2}$.



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Theorem 6.12 (Ratio test)**.** Let $\sum_{k=0}^{n} a_k$ be a series. If there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| < 1$$

the series is absolute convergent. If there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| > 1$$

the series is divergent. I.e. for

$$q = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

if $q < 1$ the series is absolute convergent, if $q > 1$ the series is divergent and if $q = 1$ the ratio test does not give evidence on convergence. ◁

Example 6.16. Does the series $\sum_{k=0}^{\infty} a_k$ with $a_k = z^{2k}$, $z \in \mathbb{K}$ converge? We apply the ratio test:

$$\begin{aligned} q &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{z^{2n+2}}{z^{2n}} \right| \\ &= \lim_{n\to\infty} |z^2| = |z^2| \end{aligned}$$

The series is convergent for $|z| < 1$ and divergent for $|z| > 1$. For $|z| = 1$ the ratio test does not give evidence on convergence. ◁

Theorem 6.13 (Alternating series test)**.** Let (a_n) be a decreasing zero sequence with positive elements. Then the series

$$\sum_{k=0}^{\infty} (-1)^k a_k$$

is convergent. ◁

Remark: The alternating series test has been discovered by Gottfried Leibniz and, hence, is sometimes called *Leibniz test* or *Leibniz criterion*.

Example 6.17. The *harmonic series*

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

is divergent. However, since the series elements form a zero sequence, a series with the same elements but alternating signs is convergent:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \ln 2$$

◁

6.6. Problems

Problem 6.1: Solve the following expressions:

$$a = \sum_{k=1}^{5} k \qquad b = \sum_{k=1}^{4} \frac{1}{k}$$

$$c = \sum_{k=4}^{7} \frac{1}{k-3} \qquad d = \sum_{k=1}^{5} (2k-1)$$

Problem 6.2: Evaluate for the following sequences the first 7 elements ($n = 0, 1, \dots, 6$) of the sequence of partial sums (s_n):

1. (a_n) with $a_n = 1$ for $n = 0, 1, 2, \dots$
2. (b_n) with $b_0 = 0, b_n = 2n{-}1$ for $n = 1, 2, 3, \dots$
3. (c_n) with $c_n = \frac{1}{2^{n-2}}$ for $n = 0, 1, 2, \dots$
4. (d_n) with $d_n = n^2$ for $n = 0, 1, 2, \dots$

Problem 6.3: Plot the first 7 partial sums of the following series in a diagram:

$$1.\ \sum_{k=0}^{\infty} 2^{-k} \qquad 2.\ \sum_{k=1}^{\infty} 3^{-k}$$

$$3.\ \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \qquad 4.\ \sum_{k=0}^{\infty} \frac{1}{k+1}$$

Problem 6.4: With the two known series

$$\sum_{k=0}^{\infty} \frac{1}{k!} = e \qquad \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = \ln 2$$

find the limit of the following expressions:

$$a = \sum_{k=0}^{\infty} \frac{-1}{k!} \qquad b = \sum_{k=0}^{\infty} \frac{(-1)^k}{3k+3}$$

$$c = \sum_{k=0}^{\infty} \left(\frac{2}{k!} + \frac{(-1)^k}{k+1} \right) \quad d = \sum_{k=0}^{\infty} \frac{2\mathrm{j} \cdot (-1)^k}{k+1}$$

Problem 6.5: Solve the following terms:

$$a = \sum_{k=0}^{11} 2^k \qquad b = \sum_{k=0}^{11} 2^{-k}$$

$$c = \sum_{k=0}^{5} 4^k \qquad d = \sum_{k=0}^{9} 1^k$$

$$e = \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^k \qquad f = \sum_{k=0}^{\infty} 3^{-k}$$

Problem 6.6: Find out which statements are true:

1. If a sequence (a_n) is a zero sequence, then the series $\sum_{k=0}^{\infty} a_n$ is convergent.

2. If a series is absolute convergent, then it is convergent.

3. If a sequence of partial sums is convergent, then the corresponding series is convergent.

4. If a series $\sum_{k=0}^{\infty} a_n$ is convergent, then the sequence (a_n) is a zero sequence.

5. If a series is convergent, then it is absolute convergent.

6. If a series is convergent, then its sequence of partial sums is convergent.

Problem 6.7: Check the following series for convergence by comparison test:

1. $\displaystyle\sum_{k=2}^{\infty} \frac{k}{k^2-1}$ 2. $\displaystyle\sum_{k=0}^{\infty} \frac{3^k}{4^k+1}$

3. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{\sqrt{k^2+1}}$ 4. $\displaystyle\sum_{k=0}^{\infty} \frac{k}{k^3+k^2-3}$

Problem 6.8: Check the following series for convergence by root test:

1. $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{k^k}$ 2. $\displaystyle\sum_{k=1}^{\infty} \frac{k^k}{5^{2k}}$

3. $\displaystyle\sum_{k=0}^{\infty} \left(\frac{k^2+3k}{1-4k^2}\right)^k$ 4. $\displaystyle\sum_{k=0}^{\infty} \frac{k}{(-3)^k}$

Problem 6.9: Check the following series for convergence by ratio test:

1. $\displaystyle\sum_{k=0}^{\infty} \frac{5^k}{k!}$ 2. $\displaystyle\sum_{k=0}^{\infty} \frac{4^k}{2^k+3^k}$

3. $\displaystyle\sum_{k=0}^{\infty} k\, e^{-k}$ 4. $\displaystyle\sum_{k=0}^{\infty} k^3\, e^{-k}$

7. Power series

7.1. Definition of power series

Definition 7.1 (Power series)**.** Let (a_n) be a sequence, $z_0 \in \mathbb{K}$ being constant and $z \in \mathbb{K}$. We define a *power series* with

$$\sum_{k=0}^{\infty} a_k z^k$$

or

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k$$

and call z_0 *expansion point*. ◁

Remark: The expansion point $z_0 \in \mathbb{K}$ acts like an offset for z. It is sufficient to study power series without this offset knowing that we can apply the gained knowledge to the more general power series by shifting its domain by z_0.

Hence, from now on we concentrate on power series with expansion point 0.

Example 7.1. Some power series (the first is a 2nd order polynomial):

- $f_1(z) = 1 + 3z - 2z^2 + 0z^3 + 0z^4 + \dots$
- $f_2(z) = \sum_{k=0}^{\infty} z^k = 1 + z + z^2 + z^3 + \dots$
- $f_3(z) = \sum_{k=0}^{\infty} k! z^k = 1 + z + 2! z^2 + 3! z^3 + \dots$
- $f_4(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots = e^z$
- $f_5(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} z^k = 1 - \frac{z}{2} + \frac{z^2}{3} - \dots$
- $f_6(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}} z^k = \frac{1}{2} - \frac{z}{4} + \frac{z^2}{8} - \dots$

◁

7.2. Convergence of power series

Definition 7.2 (Convergence)**.** If a power series has a limit we call it *convergent* – and *divergent* otherwise. For all $z \in \mathbb{K}$ where the power series is convergent we may treat the power series as a function and write:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

◁

Remark: How do we find out whether a power series is convergent? With

$$\sum_{k=0}^{\infty} a_k z^k$$

we apply the ratio test:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1} z^{n+1}}{a_n z^n} \right| &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} z \right| \\ &= |z| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= |z| q < 1 \end{aligned}$$

where q is the ratio to check convergence of the series $\sum_{k=0}^{\infty} a_k$:

$$q = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

If $q = 0$ then the power series is absolute convergent for any $z \in \mathbb{K}$. If $q = \infty$ the power series is convergent for $z = 0$ only. If $0 < q < \infty$ then the power series is:

absolute convergent	for	$\lvert z\rvert < \frac{1}{q}$
divergent	for	$\lvert z\rvert > \frac{1}{q}$
unclear	for	$\lvert z\rvert = \frac{1}{q}$

(The same may be done with the root test.)

This leads us to the *main theorem of power series*:

Theorem 7.3 (Main theorem on power series)**.** Let (a_n) be a sequence and $z \in \mathbb{K}$. For the power series

$$\sum_{k=0}^{\infty} a_k z^k$$

one of the following holds:

1. convergence for $z = 0$ only
2. convergence for all $z \in \mathbb{K}$
3. There is a number $r \in \mathbb{R}_{>0}$ such that the power series is absolute convergent for all $|z| < r$ and divergent for all $|z| > r$. For $|z| = r$ convergence in unclear.

This number r is called *radius of convergence*. If the limit exists it may be calculated by:

$$r = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

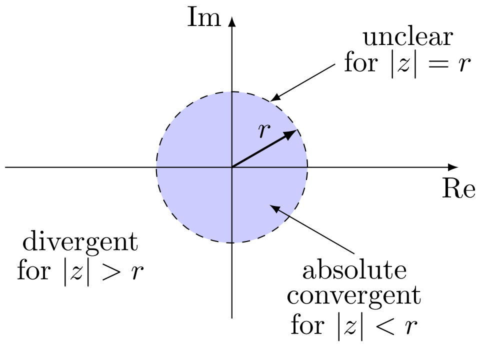
or

$$r = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}}$$

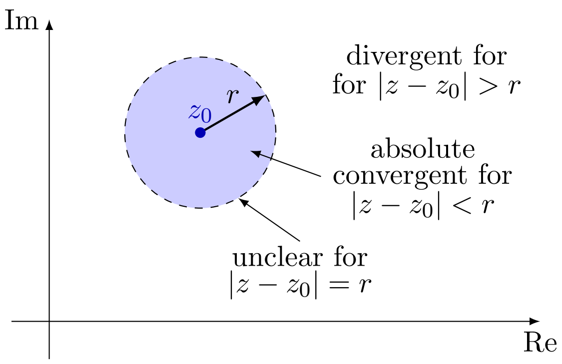
◁

Remark: Please mind that the radius of convergence r in this chapter differs from the q to check convergence of series in the previous chapter!

Remark: On the complex plane the set of all z where the power series is convergent represents a circular disc. The power series is convergent for all z within this disc and divergent for all z outside this disc. Along the edge of this disc convergence is unclear.



For a power series with expansion point z_0 (see definition 7.1) the disc shifts with its centre to z_0:



Example 7.2. We look at the power series in example 7.1:

- $f_1(z)$ is absolute convergent for all $z \in \mathbb{C}$. (A finite polynomial is convergent for all $z \in \mathbb{C}$.)
- $f_2(z)$ is absolute convergent for all $|z| < 1$ (ratio or root test).
- $f_3(z)$ is absolute convergent for $z = 0$ only (ratio or root test).
- $f_4(z)$ is absolute convergent for all $z \in \mathbb{C}$ (ratio or root test).
- $f_5(z)$ is absolute convergent for all $|z| < 1$ (root test).
- $f_6(z)$ is absolute convergent for all $|z| < 2$ (root test).

◁

7.3. Exponential function

Definition 7.4 (Exponential function)**.** With $z \in \mathbb{K}$ we define the *exponential function* with:

$$\begin{aligned} \exp(z) &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \\ &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots \end{aligned}$$

◁

Theorem 7.5 (Calculating with exponential function)**.** With $e \in \mathbb{R}$, $e = \exp(1)$ and $z \in \mathbb{K}$ we have:

$$e^z = \exp(z)$$

With $z_1, z_2 \in \mathbb{K}$ we have:

$$e^{z_1 + z_2} = e^{z_1} \cdot e^{z_2}$$

◁

Remark: Especially for $z \in \mathbb{C}$, $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $z = a + \mathrm{j}b$ we have

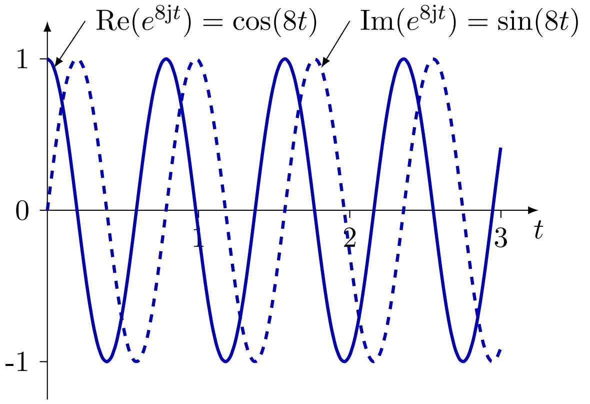
$$e^z = e^a \cdot e^{\mathrm{j}b}$$

where e^a is a damping factor over $-a$ and $e^{\mathrm{j}b}$ is oscillating over b with amplitude one.

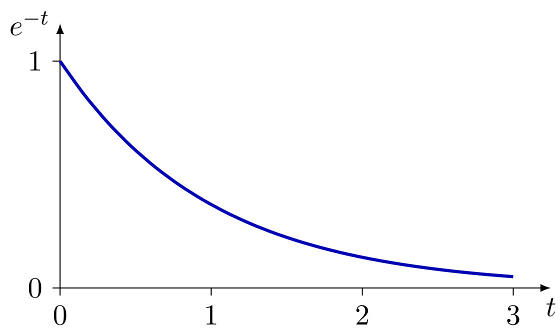
Example 7.3. We want to analyse the function

$$f(t) = e^{(-1+8\mathrm{j})t} = e^{-t} \cdot e^{8\mathrm{j}t}$$

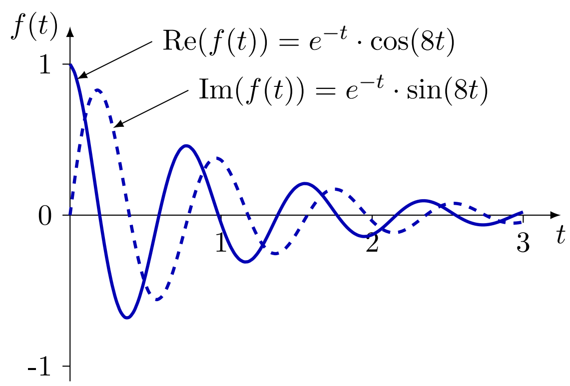
for $t \in \mathbb{R}$, $0 \leq t \leq 3$. The imaginary part of the exponent results in an oscillating value. For $0 \leq t \leq 3$ the imaginary part of the exponent ranges from 0 to 24 which equals almost four cycles ($\frac{24}{2\pi} \approx 3.82$).



The real part of the exponent is a damping factor from e^0 to e^{-3}:



The function $f(t)$ is the product of the two exponential functions:



In engineering the real part is often used to express a damped oscillator. ◁

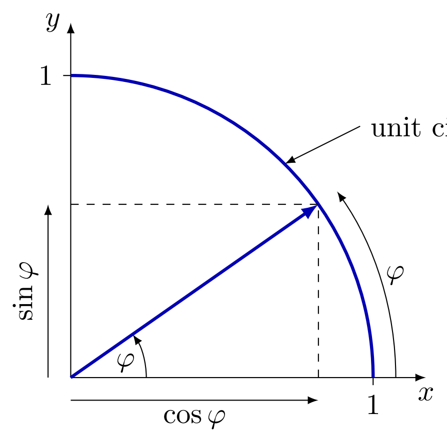
7.4. Trigonometric functions

Definition 7.6 (Sine and cosine function)**.** With $z \in \mathbb{K}$ we define the sine and cosine function with:

$$\begin{aligned}
\sin(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} \\
&= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \dots \\
\cos(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} \\
&= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \dots
\end{aligned}$$

◁

Remark: The geometric representation of sine and cosine for real arguments on a unit circle are given below:



Remark: Due to the odd exponents of z the sine is point symmetric

$$\sin(z) = -\sin(-z)$$

whereas the cosine with its even powers is axial symmetric:

$$\cos(z) = \cos(-z)$$

Theorem 7.7 (Euler's formula)**.** For $z \in \mathbb{K}$ we have

$$e^{\mathrm{j}z} = \cos z + \mathrm{j}\sin z$$

◁

Proof.

$$e^{\mathrm{j}z} = \sum_{k=0}^{\infty} \frac{(\mathrm{j}z)^k}{k!}$$

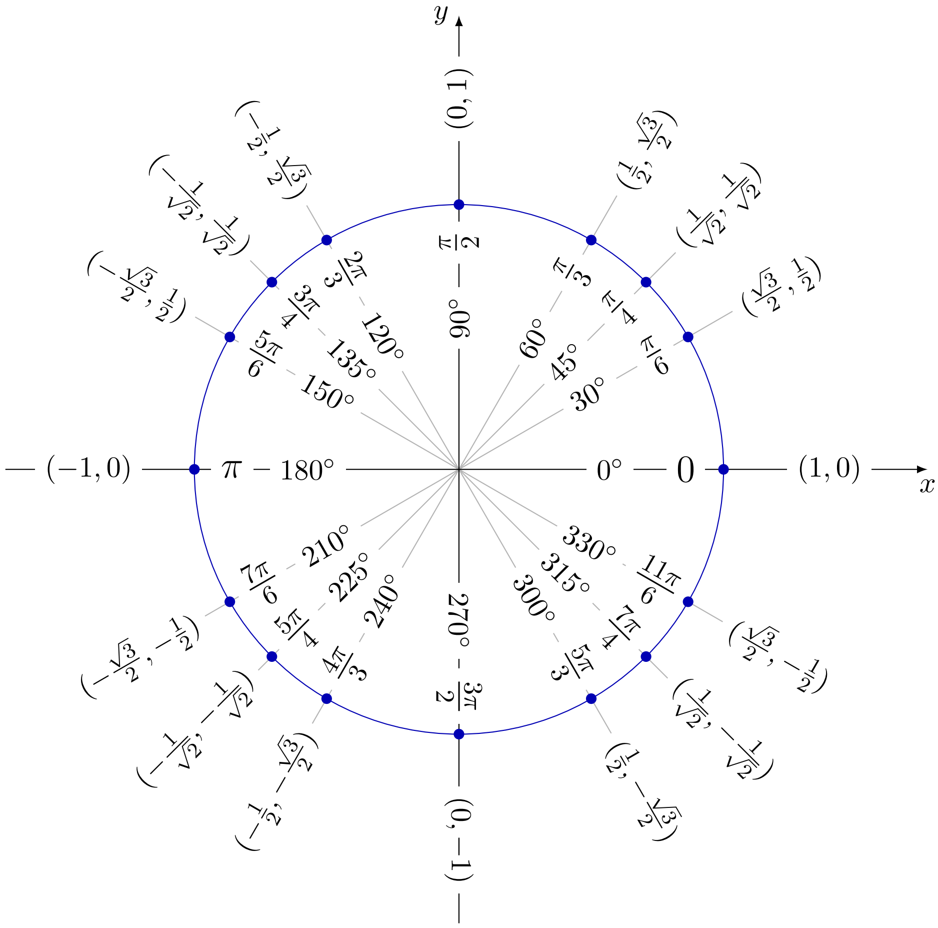


Figure 7.1.: Coordinates for some angles on a unit circle, i.e. $(\cos\varphi, \sin\varphi)$, $\varphi \in \mathbb{R}$.

$$
\begin{aligned}
&= 1 + \mathrm{j}z + \frac{(\mathrm{j}z)^2}{2!} + \frac{(\mathrm{j}z)^3}{3!} + \frac{(\mathrm{j}z)^4}{4!} + \dots \\
&= 1 + \mathrm{j}z - \frac{z^2}{2!} - \mathrm{j}\frac{z^3}{3!} + \frac{z^4}{4!} + \mathrm{j}\frac{z^5}{5!} - \dots \\
&= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots\right) \\
&\quad + \mathrm{j}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\right) \\
&= \cos z + \mathrm{j}\sin z
\end{aligned}
$$

□

Remark: Euler's formula is of particular interest for real arguments:

$$e^{\mathrm{j}x} = \cos x + \mathrm{j}\sin x \qquad \text{with } x \in \mathbb{R}$$

Remark: Due to the symmetry of the sine and cosine function we have for a negative exponent:

$$e^{-\mathrm{j}z} = \cos(-z) + \mathrm{j}\sin(-z) = \cos z - \mathrm{j}\sin z$$

This leads us to the weighted sums of exponential functions to express the sine and cosine function:

Theorem 7.8 (Sine and cosine by exponential function)**.** With $z \in \mathbb{K}$ we have

$$
\begin{aligned}
\sin z &= \frac{e^{\mathrm{j}z} - e^{-\mathrm{j}z}}{2\mathrm{j}} \\
\cos z &= \frac{e^{\mathrm{j}z} + e^{-\mathrm{j}z}}{2}
\end{aligned}
$$

◁

Remark: Now we can derive the addition theorem for sine and cosine, e.g.

$$
\begin{aligned}
2\mathrm{j}\sin(a+b) &= e^{\mathrm{j}(a+b)} - e^{-\mathrm{j}(a+b)} \\
&= e^{\mathrm{j}a}e^{\mathrm{j}b} - e^{-\mathrm{j}a}e^{-\mathrm{j}b} \\
&= (\cos a + \mathrm{j}\sin a)(\cos b + \mathrm{j}\sin b) \\
&\quad - (\cos a - \mathrm{j}\sin a)(\cos b - \mathrm{j}\sin b) \\
&= \cos a\cos b + \mathrm{j}\cos a\sin b \\
&\quad + \mathrm{j}\sin a\cos b - \sin a\sin b \\
&\quad - \cos a\cos b + \mathrm{j}\cos a\sin b \\
&\quad + \mathrm{j}\sin a\cos b + \sin a\sin b \\
&= 2\mathrm{j}\cos a\sin b + 2\mathrm{j}\sin a\cos b \\
\sin(a+b) &= \cos a\sin b + \sin a\cos b
\end{aligned}
$$

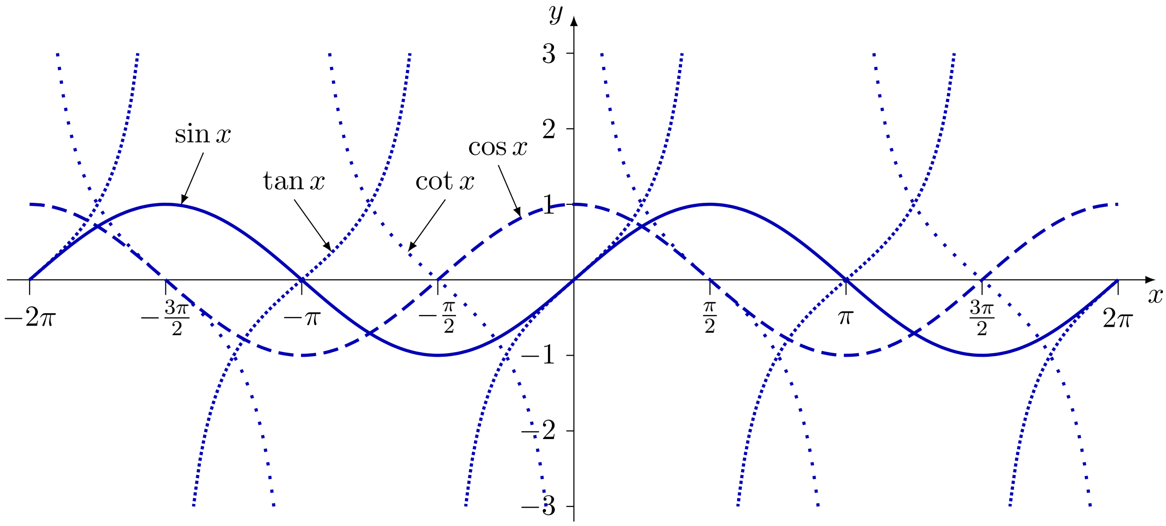


Figure 7.2.: Sine, cosine, tangent and cotangent for real arguments in the range $-2\pi \ldots 2\pi$.

Theorem 7.9 (Addition theorem)**.** With $a, b \in \mathbb{K}$ we have:

$$\sin(a \pm b) = \sin a \cdot \cos b \pm \cos a \cdot \sin b$$
$$\cos(a \pm b) = \cos a \cdot \cos b \mp \sin a \cdot \sin b$$

◁

Definition 7.10 (Tangent, cotangent)**.** With $z \in \mathbb{K}$ and $\cos(z) \neq 0$ we define:

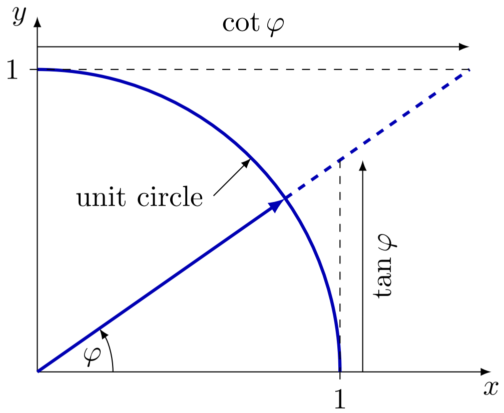
$$\tan(z) = \frac{\sin(z)}{\cos(z)}$$

With $z \in \mathbb{K}$ and $\sin(z) \neq 0$ we define:

$$\cot(z) = \frac{\cos(z)}{\sin(z)}$$

◁

Remark: The geometric representation of tangent and cotangent for real arguments on a unit circle are given below:



Remark: Some easy to memorize values for sine, cosine, tangent and cotangent for real angles:

angle in deg	rad	sin	cos	tan	cot
0	0	$\frac{\sqrt{0}}{2} = 0$	$\frac{\sqrt{4}}{2} = 1$	0	undef.
30	$\frac{\pi}{6}$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$	$\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$	1	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$
90	$\frac{\pi}{2}$	$\frac{\sqrt{4}}{2} = 1$	$\frac{\sqrt{0}}{2} = 0$	undef.	0

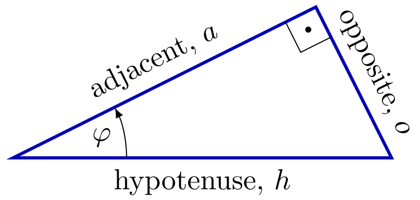
Due to symmetry and periodicity we can derive the function values for all these angles plus $\frac{n\pi}{2}$, $n \in \mathbb{Z}$. E.g. for the sine function we have:

$$\sin\varphi = -\sin(-\varphi)$$
$$\sin\varphi = -\sin(\varphi + \pi)$$
$$\sin\varphi = \sin(\varphi + 2n\pi), \quad n \in \mathbb{Z}$$

Similar relations (but not exactly the same) hold for the cosine, tangent and cotangent functions.

Remark: In an right-angled triangle we have:

$$\sin(\varphi) = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\cos(\varphi) = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\tan(\varphi) = \frac{\text{opposite}}{\text{adjacent}}$$
$$\cot(\varphi) = \frac{\text{adjacent}}{\text{opposite}}$$



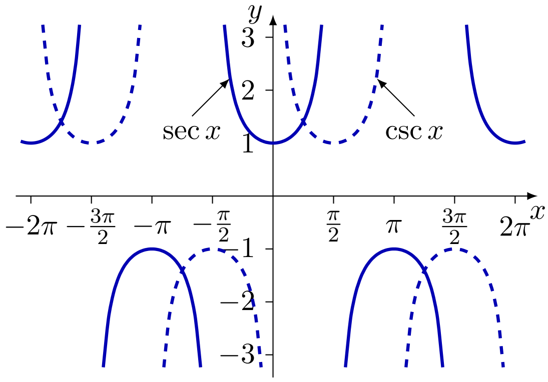
Definition 7.11 (Secant and cosecant)**.** With $\varphi \in \mathbb{K}$ and $\cos(\varphi) \neq 0$ we define the *secant* as:

$$\sec(\varphi) = \frac{1}{\cos(\varphi)}$$

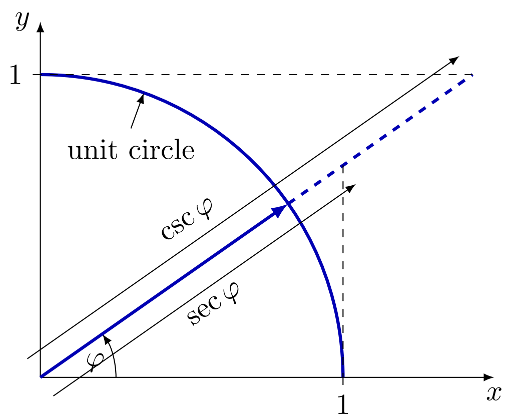
With $\varphi \in \mathbb{K}$ and $\sin(\varphi) \neq 0$ we define the *cosecant* as:

$$\csc(\varphi) = \operatorname{cosec}(\varphi) = \frac{1}{\sin(\varphi)}$$

◁



Remark: The geometric representation of secant and cosecant for real arguments on a unit circle are given below:



Definition 7.12 (Inverse trigonometric functions)**.** We define the inverse of sin, cos, tan, cot, sec and csc with real arguments as:

$$\arcsin : \begin{cases} [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}] \\ \sin(y) \mapsto y \end{cases}$$

$$\arccos : \begin{cases} [-1,1] \to [0,\pi] \\ \cos(y) \mapsto y \end{cases}$$

$$\arctan : \begin{cases} (-\infty,\infty) \to (-\frac{\pi}{2}, \frac{\pi}{2}) \\ \tan(y) \mapsto y \end{cases}$$

$$\operatorname{arccot} : \begin{cases} (-\infty,\infty) \to (0,\pi) \\ \cot(y) \mapsto y \end{cases}$$

$$\operatorname{arcsec} : \begin{cases} (-\infty,-1] \cup [1,\infty) \to (0,\pi) \setminus \{\frac{\pi}{2}\} \\ \sec(y) \mapsto y \end{cases}$$

$$\operatorname{arccsc} : \begin{cases} (-\infty,-1] \cup [1,\infty) \to (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus \{0\} \\ \csc(y) \mapsto y \end{cases}$$

◁

Remark: Other notations for the inverse trigonometric functions are $\sin^{-1}(x)$, $\cos^{-1}(x)$ etc. Do not confuse this notation with the reciprocal, e.g.

$$\arcsin(x) = \sin^{-1}(x) \neq (\sin(x))^{-1} = \frac{1}{\sin(x)}$$

7.5. Problems

Problem 7.1: Which of the following series are power series?

$$f_1(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k+1}$$

$$f_2(z) = \sum_{k=0}^{\infty} (z^k + z^{-k})$$

$$f_3(z) = \sum_{k=0}^{\infty} \frac{(z+\sqrt{z})^k}{k+1}$$

$$f_4(z) = \sum_{k=2}^{\infty} \frac{(z+2)^k}{z^2+4z+4}$$

Problem 7.2: Find the radius of convergence for the following power series:

$$f_1(z) = \sum_{k=0}^{\infty} z^k$$

$$f_2(z) = \sum_{k=0}^{\infty} \frac{z^k}{3^k}$$

$$f_3(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k^2} z^k$$

$$f_4(z) = \sum_{k=0}^{\infty} (-2z+2\mathrm{j})^k$$

$$f_5(z) = \sum_{k=0}^{\infty} \frac{(4z-4)^{k+1}(z+1)}{z^2-1}$$

Problem 7.3: Plot the region of convergence on the complex plane for $f_4(z)$ and $f_5(z)$ of the previous problem.

Problem 7.4: Find the radius of convergence for the exponential function.

Problem 7.5: For $t \in \mathbb{R}$ plot the following functions in the range $0 \leq t \leq \pi$:

$$f_1(t) = e^{t/\pi} \qquad f_2(t) = e^{-2t/\pi}$$
$$f_3(t) = e^{2\mathrm{j}t} \qquad f_4(t) = e^{-3\mathrm{j}t}$$
$$f_5(t) = e^{(1/\pi+10\mathrm{j})t} \qquad f_6(t) = e^{-(1/\pi+6\mathrm{j})t}$$

Problem 7.6: Show with Euler's formula that the following holds for any $z \in \mathbb{C}$:

$$\sin^2 z + \cos^2 z = 1$$

Problem 7.7: Show that the following equations hold:

$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

Problem 7.8: Express the following terms by exponential functions only and try to reduce them to a minimum:

$$f_1(\varphi) = \cos\varphi - \mathrm{j}\sin\varphi$$
$$f_2(\varphi) = \sin\varphi + \mathrm{j}\cos\varphi$$
$$f_3(\varphi) = 2\mathrm{j}\sin\varphi + e^{-\mathrm{j}\varphi}$$
$$f_4(\varphi) = \cos\varphi - \tfrac{1}{2}e^{\mathrm{j}\varphi}$$
$$f_5(\varphi) = \tfrac{1}{2}\mathrm{Re}(e^{\mathrm{j}\varphi}) + \frac{\mathrm{j}\sin\varphi}{2}$$
$$f_6(\varphi) = \cos^2\varphi + \mathrm{Im}(e^{\mathrm{j}\varphi})\sin\varphi$$

Problem 7.9: Simplify the following terms (without a pocket calculator):

$$x_1 = \sin\tfrac{\pi}{4}\cos\tfrac{\pi}{4} \qquad x_2 = \tan\tfrac{\pi}{4}\cot\tfrac{\pi}{4}$$
$$x_3 = \tan 1 \cot 1 \qquad x_4 = \frac{\cot 2}{\cos 2}\sin 2$$
$$x_5 = \cos\tfrac{\pi}{6}\tan\tfrac{\pi}{3} \qquad x_6 = \cot\tfrac{\pi}{3}\cos\tfrac{\pi}{6}$$

8. Functions

8.1. Definition of functions

In chapter 1 we saw already a broad definition for functions, i.e. there domain and image where arbitrary sets. In calculus we concentrate mainly on real and complex numbers and, hence, domain and image are subsets of real or complex numbers.

Definition 8.1 (Function)**.** Let $D \subseteq \mathbb{K}$ and $x \in D$.

- A *real-valued function* maps every element of D to exactly one element of $\mathbb{R}$ and we write:
$$f : \begin{cases} D \to \mathbb{R} \\ x \mapsto f(x) \end{cases}$$
- A *complex-valued function* maps every element of D to exactly one element of $\mathbb{C}$ and we write:
$$f : \begin{cases} D \to \mathbb{C} \\ x \mapsto f(x) \end{cases}$$

$\triangleleft$

Example 8.1.

- $f : \begin{cases} \mathbb{C} \to \mathbb{R} \\ x \mapsto c \end{cases}$ for $c \in \mathbb{R}$

  real-valued *constant function*.
- $\mathrm{id}_{\mathbb{C}} : \begin{cases} \mathbb{C} \to \mathbb{C} \\ x \mapsto x \end{cases}$

  complex-valued *identity function*.
- $\mathrm{abs} : \begin{cases} \mathbb{C} \to \mathbb{R} \\ x \mapsto |x| = \sqrt{\mathrm{Re}^2(x) + \mathrm{Im}^2(x)} \end{cases}$

  real-valued *absolute function*.
- $p : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto p(x) = \sum_{k=0}^{n} a_k x^k \end{cases}$

  real-valued polynomial of n^{th} order with coefficients $a_0, a_1, \ldots, a_n \in \mathbb{R}$.
- $\exp : \begin{cases} \mathbb{C} \to \mathbb{C} \\ x \mapsto \exp(x) \end{cases}$

  complex-valued exponential function.

$\triangleleft$

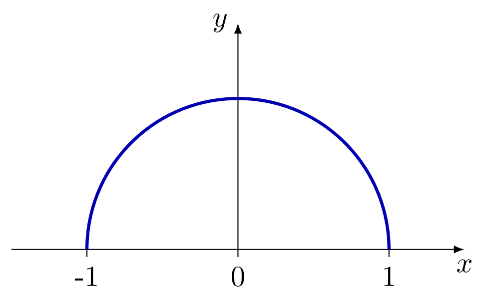
Remark: There are different ways to describe a function:

1. *analytical* either in *implicit* form $f(x,y) = 0$, e.g.
$$x^2 + y^2 - 1 = 0$$
   or in *explicit* form $y = f(x)$, e.g.
$$y = \sqrt{1 - x^2}$$
2. by a table, e.g.

x	y
-1.0	0.000
-0.8	0.600
-0.6	0.800
-0.4	0.917
-0.2	0.980
0.0	1.000
0.2	0.980
0.4	0.917
0.6	0.800
0.8	0.600
1.0	0.000

3. as a set, e.g.
$$\{(x,y) \in [-1,1] \times \mathbb{R} \mid y = \sqrt{1-x^2}\}$$
4. with parameter, e.g.
$$x = \cos t, \quad y = \sin t \quad \text{for } t \in [0, \pi]$$

The examples describe the function illustrated by the following graph:



8.2. Properties of functions

Definition 8.2 (Monotonicity)**.** A real-valued function $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}$ and $a, b \in D$ with $a > b$ is called:

- *increasing* if $f(a) \geq f(b)$
- *decreasing* if $f(a) \leq f(b)$
- *strictly increasing* if $f(a) > f(b)$
- *strictly decreasing* if $f(a) < f(b)$

If a function is (strictly) increasing or (strictly) decreasing we call it (strictly) *monotone*. ◁

Definition 8.3 (Bounded function)**.** Let $D \subseteq \mathbb{K}$ and $f : D \to \mathbb{K}$ be a real- or complex-valued function over D. We say f is bounded if there exist an $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in D$:

$$\exists M \in \mathbb{R}\ \forall x \in D : |f(x)| \leq M$$

◁

Definition 8.4 (Symmetry)**.** Let $D \subseteq \mathbb{K}$ be the domain of a real- or complex-valued function $f : D \to \mathbb{K}$. The function f shows

- *reflection symmetry* if $f(-x) = f(x)$
- *point symmetry* if $f(-x) = -f(x)$

◁

Remark: The definition of point symmetry may be extended to point symmetry w.r.t. a point (x_0, y_0). Similarly reflection symmetry may be extended to reflection symmetry w.r.t. a line defined by $ax + by = c$.

Definition 8.5 (Periodicity)**.** Let $D \subseteq \mathbb{K}$ be the domain of a real- or complex-valued function $f : D \to \mathbb{K}$. The function f is said to be periodic if there exist a non-zero constant $T \in D$ with

$$f(x + T) = f(x) \qquad \text{for all } x + T, x \in D$$

◁

Example 8.2.

- $\exp : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \exp(x) \end{cases}$

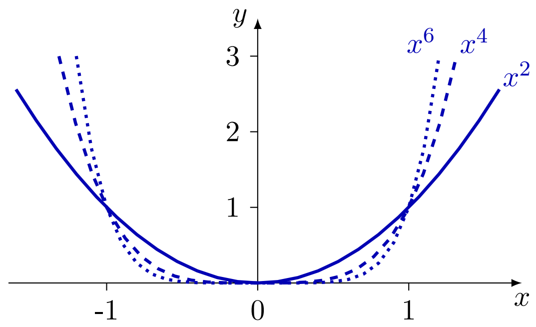
  is strictly increasing, is not bounded, is not symmetric and is not periodic.

- $\sin : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \sin(x) \end{cases}$

  is not monotone, is bounded, is point symmetric and is periodic over 2π.

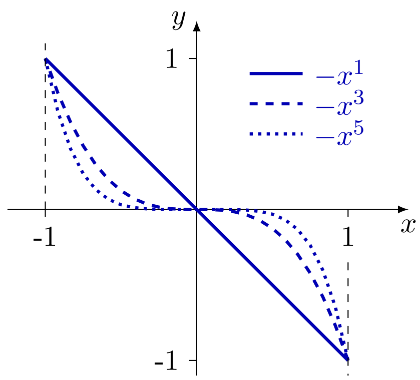
- $p : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^{2n}, n \in \mathbb{N} \end{cases}$

  is not monotone, is not bounded, shows reflection symmetry and is not periodic.



- $p : \begin{cases} [-1, 1] \to \mathbb{R} \\ x \mapsto -x^{2n-1}, n \in \mathbb{N} \end{cases}$

  is strictly decreasing, is bounded, shows point symmetry and is not periodic.



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Definition 8.6 (Zero of a function)**.** Let $D \subseteq \mathbb{K}$ be the domain of a real- or complex-valued function $f : D \to \mathbb{K}$ and $x' \in D$. For

$$f(x') = 0$$

we say x' is a *zero* of f. ◁

Definition 8.7 (Extrema)**.** Let $D \subseteq \mathbb{K}$ be the domain of a real-valued function $f : D \to \mathbb{K}$ with $x' \in D$ and the epsilon neighbourhood $U_\varepsilon(x')$. We say x' is a

- *local maximum* if

$$f(x') \geq f(x) \qquad \text{for all } x \in U_\varepsilon(x') \cap D$$

- *local minimum* if

$$f(x') \le f(x) \qquad \text{for all } x \in U_\varepsilon(x') \cap D$$

- *global maximum* if

$$f(x') \ge f(x) \qquad \text{for all } x \in D$$

- *global minimum* if

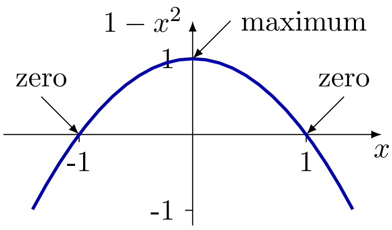
$$f(x') \le f(x) \qquad \text{for all } x \in D$$

If x' is a global/local maximum or minimum we call it global/local *extremum*. ◁

Example 8.3.

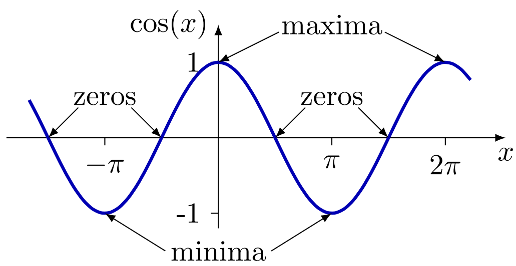
- $p : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto 1 - x^2 \end{cases}$

  has zeros at $x = \pm 1$ and a global maximum at $x = 0$.



- $\cos : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \cos(x) \end{cases}$

  has zeros at $x = n\pi + \frac{\pi}{2}$, maxima at $x = 2n\pi$ and minima at $x = 2n\pi + \pi$ for $n \in \mathbb{Z}$.



- $p : \begin{cases} \mathbb{C} \to \mathbb{C} \\ x \mapsto x^2 + 1 \end{cases}$

  has zeros at $x = \pm \mathrm{j}$. (Extrema are not defined for complex-valued functions.)

- $\ln : \begin{cases} \mathbb{R}_{>0} \to \mathbb{R} \\ x \mapsto \ln(x) \end{cases}$

  has a zero at $x = 1$ and no extremum.

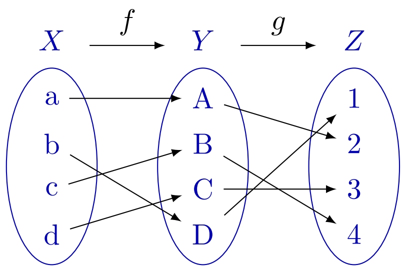
◁

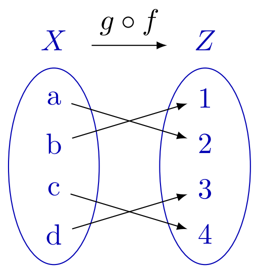
8.3. Composed and inverse functions

Definition 8.8 (Function composition)**.** Let X, Y and Z be sets and $f : X \to Y$ and $g : Y \to Z$ be two functions. We define

$$g \circ f : \begin{cases} X \to Z \\ x \mapsto g(f(x)) \end{cases}$$

as the *composition* of f and g. ◁





Remark: For the composition commutativity does not hold, i.e. $g \circ f \ne f \circ g$. The functions must be applied from right to left, i.e. for $(g \circ f)(x)$ the function f is applied first and its value serves as input for g, i.e. $g(f(x))$.

Example 8.4.

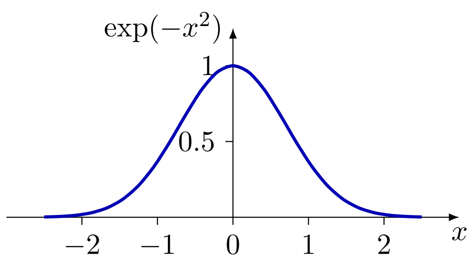
$$g \circ f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \exp(-x^2) \end{cases}$$

is the composition of

$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto -x^2 \end{cases}$$

and

$$g : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \exp(x) \end{cases}$$



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Definition 8.9 (Inverse function)**.** Let X and Y be sets and $f : X \to Y$ be a bijective function. We define

$$f^{-1} : \begin{cases} Y \to X \\ f(x) \mapsto x \end{cases}$$

as the *inverse function*. ◁

Remark: Please mind: The inverse is not the reciprocal:

$$f^{-1} \neq \frac{1}{f(x)}$$

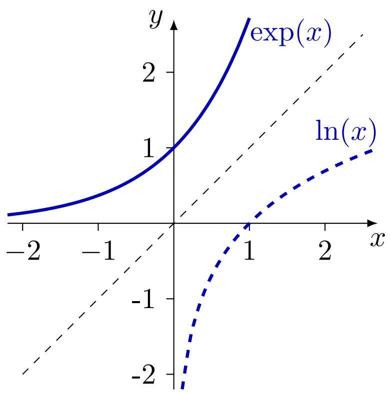
In a Cartesian diagram the inverse is the graph reflected at the diagonal line $x = y$.

Example 8.5.

- $\exp : \begin{cases} \mathbb{R} \to \mathbb{R}_{>0} \\ x \mapsto \exp(x) \end{cases}$

  has the inverse:

  $\ln : \begin{cases} \mathbb{R}_{>0} \to \mathbb{R} \\ x \mapsto \ln(x) \end{cases}$



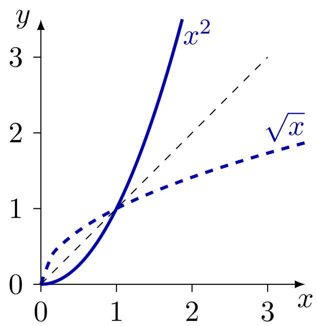
- $p : \begin{cases} \mathbb{R} \to \mathbb{R}_{\geq 0} \\ x \mapsto x^2 \end{cases}$

  has no inverse since it is not bijective.

- $p : \begin{cases} \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \\ x \mapsto x^2 \end{cases}$

  has the inverse:

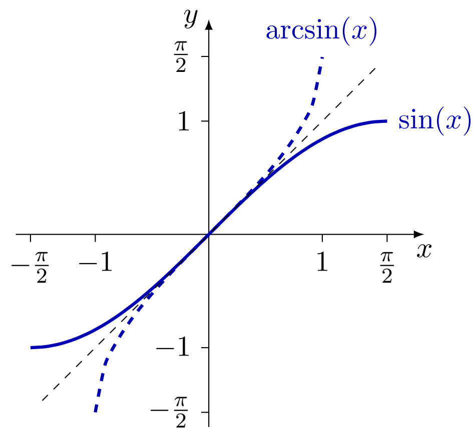
  $\mathrm{sqrt} : \begin{cases} \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \\ x \mapsto \sqrt{x} \end{cases}$



- $\sin : \begin{cases} [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1] \\ x \mapsto \sin(x) \end{cases}$

  has the inverse:

  $\arcsin : \begin{cases} [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}] \\ x \mapsto \arcsin(x) \end{cases}$



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8.4. Continuity

For continuity we limit ourselves to real domains here. For a complex domain the rules of multiple argument functions apply which we will handle in Math 2.

Definition 8.10 (Limit of a function)**.** Let $D \subseteq \mathbb{R}$ be the domain of a real- or complex-valued function $f : D \to \mathbb{K}$. With $x_0 \in D$, $x \in D \setminus \{x_0\}$ and $y_0 \in \mathbb{R}$ or $y_0 \in \mathbb{C}$ the expression

$$\lim_{x \to x_0} f(x) = y_0$$

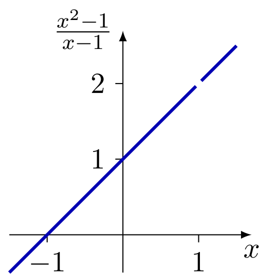
means that $f(x)$ can be as close as desired to y_0 by making x sufficient close to x_0. ◁

Example 8.6. The function

$$f : \begin{cases} \mathbb{R} \setminus \{1\} \to \mathbb{R} \\ x \mapsto \frac{x^2 - 1}{x - 1} \end{cases}$$

is not defined for $x = 1$. However the limit is:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$



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Definition 8.11 (One-sided limit)**.** Let $D \subseteq \mathbb{R}$ be the domain of a real- or complex-valued function $f : D \to \mathbb{K}$, $x_0 \in D$ and $x \in D \setminus \{x_0\}$. We define the

- *left-sided limit* as the limit of $f(x)$ with $x < x_0$:
$$\lim_{x \to x_0^-} f(x)$$
- *right-sided limit* as the limit of $f(x)$ with $x > x_0$:
$$\lim_{x \to x_0^+} f(x)$$

◁

Example 8.7. The *sign function*

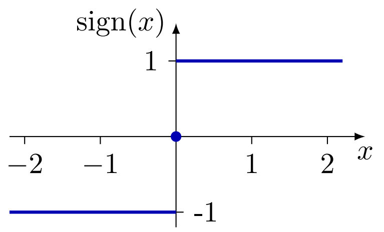
$$\text{sign} : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto 1 & \text{for } x > 0 \\ x \mapsto 0 & \text{for } x = 0 \\ x \mapsto -1 & \text{for } x < 0 \end{cases}$$

has the right-sided limit

$$\lim_{x \to 0^+} \text{sign}(x) = 1$$

and the left-sided limit

$$\lim_{x \to 0^-} \text{sign}(x) = -1$$



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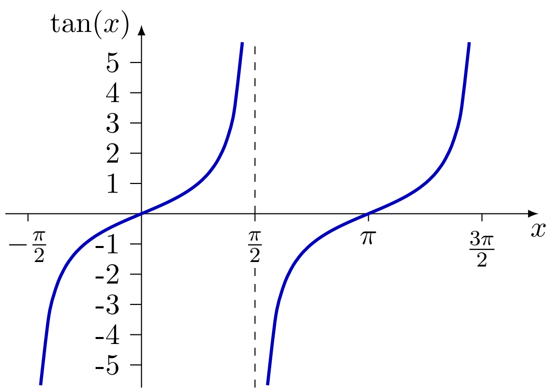
Example 8.8. We analyse the tangent function at $\frac{\pi}{2}$ from the left and the right:

$$\lim_{x \to \frac{\pi}{2}^-} \tan(x) = \infty$$
$$\lim_{x \to \frac{\pi}{2}^+} \tan(x) = -\infty$$



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Definition 8.12 (Continuity)**.** Let $D \subseteq \mathbb{R}$ be the domain of a real- or complex-valued function $f : D \to \mathbb{K}$, $x_0 \in D$ and $x \in D \setminus \{x_0\}$. We say f is *continuous* in x_0 if

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0)$$

If f is continuous for all $x_0 \in D$ we say f is a *continuous function.* ◁

Remark: Visually speaking, if you are able to draw a function without lifting the pen it is continuous.

Example 8.9.

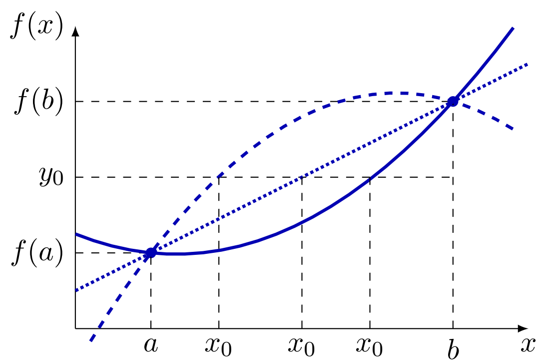
- $\exp(x)$, $\sin(x)$ and $\cos(x)$ are continuous functions in $\mathbb{R}$.
- $\tan(x)$ is continuous in $(-\frac{\pi}{2}, \frac{\pi}{2})$.
- $\text{sign}(x)$ (see previous example) is not continuous at $x = 0$.
- All polynomials are continuous.

◁

Theorem 8.13 (Intermediate value theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be a real-valued continuous function over $[a, b]$, $a < b$ and $f(a) < f(b)$. For any value $y_0 \in [f(a), f(b)]$ there exists an $x_0 \in [a, b]$ with $f(x_0) = y_0$. In short

$$\forall y_0 \in [f(a), f(b)]\ \exists x_0 \in [a, b] : f(x_0) = y_0$$

The corresponding holds for $f(a) > f(b)$. ◁

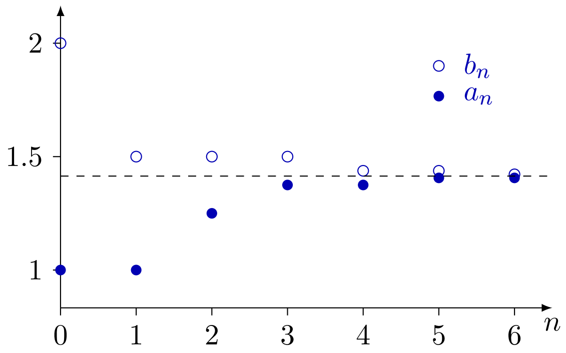


Remark: Especially for $f(a) \cdot f(b) < 0$ (i.e. one factor is positive and the other negative) the intermediate value theorem states that there must be a at least one zero between a and b.

Example 8.10. The *bisection method* is an algorithm to find zeros of an arbitrary function $f : \mathbb{R} \to \mathbb{R}$. First two values a_0 and b_0 are required where $f(a_0) \cdot f(b_0) < 0$, say, $f(a_0) < 0$ and $f(b_0) > 0$. Then the function value at $c_0 = \frac{a_0+b_0}{2}$ is determined. If $f(c_0) < 0$ then $a_1 = c_0$ and $b_1 = b_0$, else $a_1 = a_0$ and $b_1 = c_0$. With a_1 and b_1 we repeat the same process and approach the zero of f.

E.g. with the function $f(x) = x^2 - 2$ and $a = 1$ and $b = 2$ we get the following table:

n	a_n	b_n	c_n
0	1	2	1.5
1	1	1.5	1.25
2	1.25	1.5	1.375
3	1.375	1.5	1.438
4	1.375	1.438	1.406
5	1.406	1.438	1.422
6	1.406	1.422	1.414
		...	



The bisection method is simple and robust, but rather slow. In computer science it is used to derive an approximation as an input for a more efficient algorithm, e.g. Newton's method. ◁

8.5. Problems

Problem 8.1: Plot the following functions:

1. $f : \begin{cases} [0, \pi] \to \mathbb{R} \\ x \mapsto \sin(x) \end{cases}$
2.

x	0	0.2	0.4	0.6	0.8	1
$f(x)$	0	0.04	0.16	0.36	0.64	1

3. $\{(x, y) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \mid y = \sqrt{x}\}$
4. $x = \pm\sqrt{t}$, $y = \exp(-t)$ for $t \in [0, 4]$.

Problem 8.2: Give statements on monotonicity, boundary, symmetry, periodicity, zeros and extrema for the following functions:

1. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto e^{-x} \end{cases}$
2. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto e^{-x^2} \end{cases}$
3. $f : \begin{cases} \mathbb{R} \to \mathbb{C} \\ x \mapsto e^{\mathrm{j}x} \end{cases}$
4. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \mathrm{Im}(e^{\mathrm{j}x}) \end{cases}$
5. $f : \begin{cases} \mathbb{R}_{<0} \to \mathbb{R} \\ x \mapsto \ln(-x) \end{cases}$
6. $f : \begin{cases} (0, \pi) \to \mathbb{R} \\ x \mapsto \cot(x) \end{cases}$
7. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^5 + x^3 + x \end{cases}$

Problem 8.3: Find the inverse of the following functions:

1. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^3 - 1 \end{cases}$
2. $f : \begin{cases} \mathbb{R}_{\leq 0} \to \mathbb{R}_{\geq 2} \\ x \mapsto x^2 + 2 \end{cases}$
3. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \cos(x) \end{cases}$
4. $f : \begin{cases} (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R} \\ x \mapsto \tan(x) \end{cases}$

Problem 8.4: Which of the following functions are continuous in their whole domain?

1. $f : \begin{cases} (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R} \\ x \mapsto \tan(x) \end{cases}$
2. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto |x| \end{cases}$
3. The sign-function as in example 8.7
4. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto (x - 1)(x + 1) \end{cases}$

Problem 8.5: A continuous function in $\mathbb{R}$ has the values $f(1) = -1$ and $f(2) = 1$. Give a statement on zeros of f.

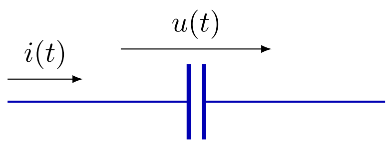
9. Differential calculus

9.1. Introduction

In engineering we are not only focussed on values at a given time or position but also in their dynamic behaviour. E.g. for speed control the police is not so much interested in the position of a car *at* a given time but rather in the speed of the car, i.e. the *change* of position *per* time.

Another example is a capacitor in electrical engineering: We investigate the change of voltage to determine the electrical current that caused the voltage change.

Let's assume a capacitor with capacity C measured in Farad, F. The capacitor is charged by a current i which is a function over time t and we write in short $i(t)$. The charging current results in a voltage over the capacitor u which also varies over time t and we write $u(t)$.



By measuring the voltage u we can evaluate the current i. If the voltage increases with constant speed over, say, two seconds by $4\,\mathrm{V}$ on a capacitor with $0.01\,\mathrm{F}$, then the charging current is

$$i = C\frac{\Delta u}{\Delta t} = 0.01\,\mathrm{F}\cdot\frac{4\,\mathrm{V}}{2\,\mathrm{s}} = 0.02\,\mathrm{A} = 20\,\mathrm{mA}$$

If the current varies over time we need to find out the current i at a given time t. To do so we reduce the time interval towards zero

$$i(t) = \lim_{\Delta t\to 0} C\frac{u(t+\Delta t) - u(t)}{\Delta t} = C\frac{\mathrm{d}u}{\mathrm{d}t}(t)$$

Let's assume the function for the voltage is a sine-wave with frequency f and amplitude $\hat{u}$ i.e.

$$u(t) = \hat{u}\sin(2\pi f t)$$

For the current $i(t)$ we get:

$$\begin{aligned} i(t) &= C\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}\sin(2\pi f t) \\ &= 2\pi\hat{u}fC\cos(2\pi f t) \\ &= \hat{\imath}\cos(2\pi f t) \end{aligned}$$

with $\hat{\imath} = 2\pi\hat{u}fC$.

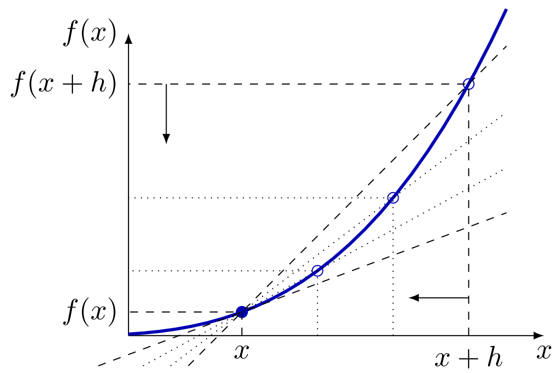
By the end of this chapter you will be able to apply the required mathematics to solve this kind of equations.

9.2. Differentiability

Definition 9.1 (Differentiability)**.** Let $D \subseteq \mathbb{R}$. We say $f : D \to \mathbb{R}$ is *differentiable* at $x \in D$ if the limit

$$\lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

exist. We call the limit the *derivative* of f at x. If f is differentiable at all $x \in D$ we say f is differentiable. ◁



Remark: With $x' = x + h$ we may express the derivative as

$$\lim_{x'\to x}\frac{f(x') - f(x)}{x' - x}$$

For the derivative of f we write $\frac{\mathrm{d}f}{\mathrm{d}x}(x)$, $\frac{\mathrm{d}}{\mathrm{d}x}f(x)$ or simply $f'(x)$. Applications in physics often deal with functions over time t and physicists write in short $\frac{\mathrm{d}f}{\mathrm{d}t}(t) = \dot{f}(t) = \dot{f}$ where the dot indicates the derivative w.r.t. time.

The x in brackets and denominator of $\frac{\mathrm{d}f}{\mathrm{d}x}(x)$ seem redundant. However, later we handle functions that depend on more than one variable. There we differentiate w.r.t. a particular variable, e.g. $\frac{\mathrm{d}}{\mathrm{d}x}f(x,y)$ or $\frac{\mathrm{d}}{\mathrm{d}y}f(x,y)$.

If the context is clear we may skip the x in brackets and write $\frac{\mathrm{d}f}{\mathrm{d}x}$.

Example 9.1.

- $\mathrm{sq} : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ is differentiable.
- $\sin : \mathbb{R} \to \mathbb{R}, x \mapsto \sin(x)$ is differentiable.
- $\mathrm{abs} : \mathbb{R} \to \mathbb{R}, x \mapsto |x|$ is not differentiable at 0.

◁

Theorem 9.2 (Differentiability implies continuity)**.** Every differentiable function is continuous. ◁

Remark: Since it is an implication it does not mean that every continuous function is differentiable, see the following example.

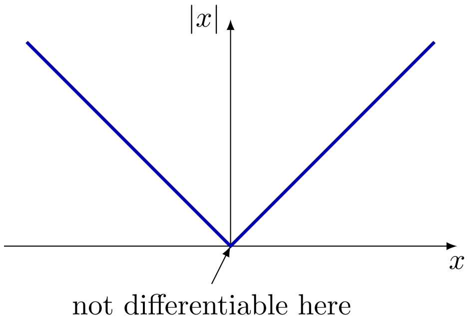
Example 9.2. Is the function of the absolute differentiable at zero?

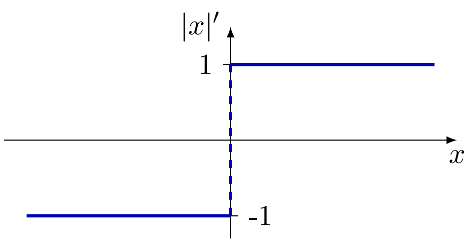
$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto |x| \end{cases}$$

The function is continuous in $\mathbb{R}$. We take the left- and right-sided limit to test differentiability at zero:

$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{|h| - |0|}{h} = -1$$
$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{|h| - |0|}{h} = +1$$

We get two different values, hence, although being continuous the function of the absolute is not differentiable at zero.





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9.3. Some derivatives

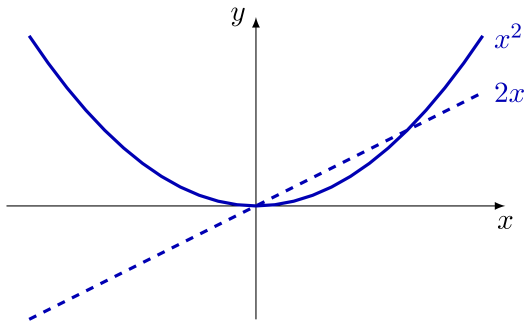
In this section we want to evaluate some typical derivatives.

Example 9.3. What is the derivative of

$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^2 \end{cases} \quad ?$$

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}x} &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \underbrace{\lim_{h\to 0} 2x}_{=2x} + \underbrace{\lim_{h\to 0} h}_{=0} = 2x
\end{aligned}$$

Hence, the derivative of x^2, $x \in \mathbb{R}$ is $2x$.



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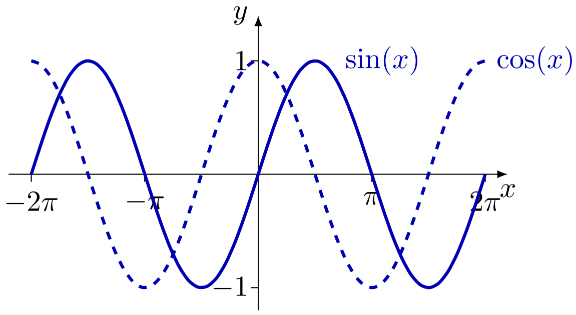
Example 9.4. What is the derivative of

$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \sin(x) \end{cases} \quad ?$$

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}x} &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\overbrace{\cos(h)}^{\to 1} + \cos(x)\overbrace{\sin(h)}^{\to h} - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x) + h\cos(x) - \sin(x)}{h} \\
&= \lim_{h\to 0} \cos(x) = \cos(x)
\end{aligned}$$

Hence, the derivative of $\sin(x)$, $x \in \mathbb{R}$ is $\cos(x)$.



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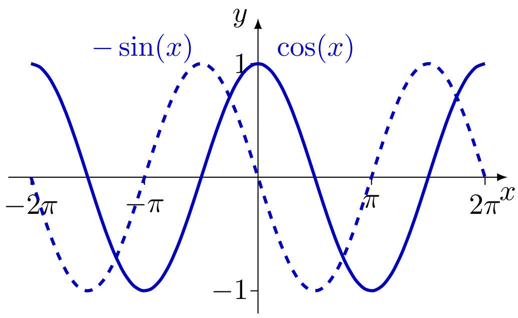
Example 9.5. What is the derivative of

$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \cos(x) \end{cases} \quad ?$$

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}x} &= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)\overbrace{\cos(h)}^{\to 1} - \sin(x)\overbrace{\sin(h)}^{\to h} - \cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x) - h\sin(x) - \cos(x)}{h} \\
&= \lim_{h\to 0} -\sin(x) = -\sin(x)
\end{aligned}$$

Hence, the derivative of $\cos(x)$, $x \in \mathbb{R}$ is $-\sin(x)$.



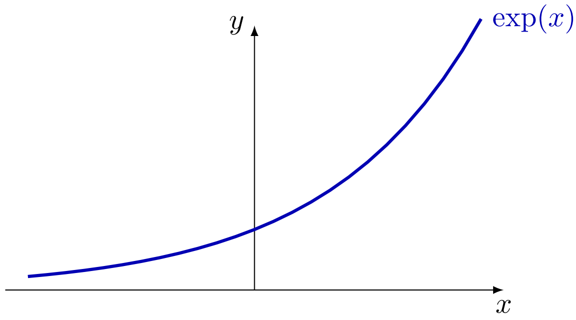
◁

Example 9.6. What is the derivative of

$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto e^x \end{cases} \quad ?$$

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}x} &= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} \\
&= \lim_{h\to 0} \frac{e^x(-1 + 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \dots)}{h} \\
&= \lim_{h\to 0} e^x \overbrace{\left(1 + \frac{h}{2!} + \frac{h^2}{3!} + \dots\right)}^{\to 1} = e^x
\end{aligned}$$

Hence, the derivative of e^x, $x \in \mathbb{R}$ is e^x.



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Some more of the *basic derivatives* are listed in the upper part of table 9.1.

$f(x)$	$f'(x)$
c	0 for $c \in \mathbb{R}$
x^c	cx^{c-1} for $c \in \mathbb{R}$
$\sqrt{x}$	$\dfrac{1}{2\sqrt{x}}$
e^x	e^x
c^x	$c^x \ln c$ for $c \in \mathbb{R}_{>0}$
$\ln\lvert x\rvert$	$\dfrac{1}{x}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$1/\cos^2(x) = 1 + \tan^2(x)$
$\cot(x)$	$-1/\sin^2(x) = -(1+\cot^2(x))$
$\arcsin(x)$	$1/\sqrt{1-x^2}$
$\arccos(x)$	$-1/\sqrt{1-x^2}$
$\arctan(x)$	$1/(1+x^2)$
$\operatorname{arccot}(x)$	$-1/(1+x^2)$
$c \cdot f(x)$	$c \cdot f'(x)$ for $c \in \mathbb{R}$
$f(x) \pm g(x)$	$f'(x) \pm g'(x)$
$f(x) \cdot g(x)$	$f'(x) \cdot g(x) + f(x) \cdot g'(x)$
$\dfrac{f(x)}{g(x)}$	$\dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$
$f(g(x))$	$f'(g) \cdot g'(x)$

Table 9.1.: Some important derivatives.

9.4. Calculation rules for derivatives

Theorem 9.3 (Basic operations on derivatives)**.** Let f and g be differentiable functions and $c \in \mathbb{R}$. Then we have

$$\begin{aligned}
(f(x) \pm g(x))' &= f'(x) \pm g'(x) \\
(c \cdot f(x))' &= c \cdot f'(x) \\
(f(x) \cdot g(x))' &= f'(x) \cdot g(x) + f(x) \cdot g'(x) \\
\left(\frac{f(x)}{g(x)}\right)' &= \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)} \\
&\text{for the last } g(x) \neq 0
\end{aligned}$$

◁

Example 9.7.

- Let $f(x) = \sin(x) + x^2$, then

$$f'(x) = \tfrac{\mathrm{d}}{\mathrm{d}x} \sin(x) + \tfrac{\mathrm{d}}{\mathrm{d}x} x^2 = \cos(x) + 2x$$

- Let $f(x) = 3x^3$, then

$$f'(x) = 3 \tfrac{\mathrm{d}}{\mathrm{d}x} x^3 = 3 \cdot 3x^2 = 9x^2$$

- Let $f(x) = \sin(x) \cdot \cos(x)$, then

$$\begin{aligned}
f'(x) &= \tfrac{\mathrm{d}\sin(x)}{\mathrm{d}x} \cos(x) + \sin(x) \tfrac{\mathrm{d}\cos(x)}{\mathrm{d}x} \\
&= \cos(x)\cos(x) + \sin(x)(-\sin(x)) \\
&= \cos^2(x) - \sin^2(x)
\end{aligned}$$

- Let $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$, then

$$\begin{aligned}
f'(x) &= \frac{\tfrac{\mathrm{d}\sin(x)}{\mathrm{d}x} \cos(x) - \sin(x) \tfrac{\mathrm{d}\cos(x)}{\mathrm{d}x}}{\cos^2(x)} \\
&= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} \\
&= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}
\end{aligned}$$

◁

Theorem 9.4 (Chain rule)**.** Let $X, Y, Z \subseteq \mathbb{R}$ and $f : Y \to Z$ and $g : X \to Y$ be two differentiable functions. With $f \circ g = f(g(x))$ as the composition of the two functions we have:

$$(f \circ g)' = (f' \circ g) \cdot g'$$

i.e.

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) = \frac{\mathrm{d}f}{\mathrm{d}g}(g) \cdot \frac{\mathrm{d}g}{\mathrm{d}x}(x)$$

◁

Remark: Mnemonic: "Inner derivative times outer derivative"

Example 9.8.

- Let $f(x) = \sin(\omega x)$, then

$$\begin{aligned}
f(g) &= \sin(g) \qquad g(x) = \omega x \\
f'(x) &= f'(g) \cdot g'(x) = \cos(g) \cdot \omega \\
&= \omega \cos(\omega x)
\end{aligned}$$

- Let $f(x) = \sin^2(x)$, then

$$\begin{aligned}
f(g) &= g^2 \qquad g(x) = \sin(x) \\
f'(x) &= f'(g) \cdot g'(x) = 2g \cdot \cos(x) \\
&= 2\sin(x)\cos(x) = \sin(2x)
\end{aligned}$$

- Let $f(x) = \sqrt{\ln|2x|}$, then

$$\begin{aligned}
f(g) &= \sqrt{g} \qquad g(h) = \ln|h| \qquad h(x) = 2x \\
f'(x) &= f'(g) \cdot g'(h) \cdot h'(x) \\
&= \frac{1}{2\sqrt{g}} \cdot \frac{1}{h} \cdot 2 = \frac{1}{2\sqrt{\ln|2x|}} \cdot \frac{1}{2x} \cdot 2 \\
&= \frac{1}{2x\sqrt{\ln|2x|}}
\end{aligned}$$

- Let $f(x) = e^{-\frac{x^2}{2}}$, then

$$f'(x) = e^{-\frac{x^2}{2}} \cdot \frac{-1}{2} \cdot 2x = -x\, e^{-\frac{x^2}{2}}$$

◁

Theorem 9.5 (Mean value theorem)**.** Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ be a differentiable function and $a, b \in I$ with $a < b$. There exist at least one $x \in I$, $a < x < b$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

◁

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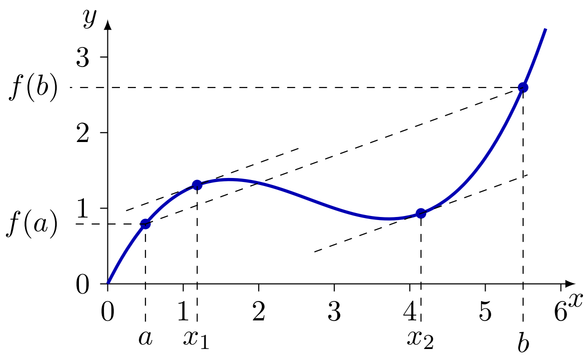
Example 9.9. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with $x \mapsto f(x) = \frac{1}{9}x^3 - \frac{8}{9}x^2 + 2x$. Let further $a = \frac{1}{2}$ and $b = 5\frac{1}{2}$ be two elements of the domain of f. Then we have:

$$\begin{aligned}
f'(x) &= \frac{f(b) - f(a)}{b - a} \\
\frac{1}{3}x^2 - \frac{16}{9}x + 2 &= \frac{2\frac{43}{72} - \frac{19}{24}}{5\frac{1}{2} - \frac{1}{2}} = \frac{13}{36} \\
x^2 - \frac{16}{3}x &= \frac{13}{12} - 6 \\
\left(x - \frac{8}{3}\right)^2 &= \frac{13}{12} - 6 + \frac{64}{9} \\
\left(x - \frac{8}{3}\right)^2 &= \frac{39}{36} - \frac{216}{36} + \frac{256}{36} = \frac{79}{36} \\
x_{1,2} &= \tfrac{8}{3} \pm \tfrac{1}{6}\sqrt{79}
\end{aligned}$$

I.e. at position x_1 and x_2 the first derivative of f equals the mean slope between a and b.



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Remark: If a derivative is differentiable it may be differentiated again. We then say *second derivative* and write $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x)$, $\frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x)$ or $f''(x)$.

If a function f is differentiated n-times we then say n^{th} derivative and we write $\frac{\mathrm{d}^n f}{\mathrm{d}x^n}(x)$, $\frac{\mathrm{d}^n}{\mathrm{d}x^n}f(x)$ or $f^{(n)}(x)$.

Example 9.10.

- Let $f(x) = x^3 - 2x^2 + 4x - 2$, then

$$\begin{aligned}
f'(x) &= 3x^2 - 4x + 4 \\
f''(x) &= 6x - 4 \\
f'''(x) &= 6 \\
f^{(4)}(x) &= 0
\end{aligned}$$

- Let $u(t) = \hat{u}\sin(\omega t)$, then

$$\begin{aligned}
u'(t) &= \omega\ \hat{u}\cos(\omega t) \\
u''(t) &= -\omega^2\,\hat{u}\sin(\omega t) \\
u'''(t) &= -\omega^3\,\hat{u}\cos(\omega t) \\
u^{(4)}(t) &= \omega^4\,\hat{u}\sin(\omega t)
\end{aligned}$$

- Let $f(t) = e^{kt}$, then

$$\begin{aligned}
f'(t) &= k\ e^{kt} \\
f''(t) &= k^2\,e^{kt} \\
f'''(t) &= k^3\,e^{kt} \\
&\dots \\
f^{(n)}(t) &= k^n\,e^{kt}
\end{aligned}$$

- Let $f(x) = \ln|x|$, then

$$\begin{aligned}
f'(x) &= \frac{1}{x} = x^{-1} \\
f''(x) &= -x^{-2} = \frac{-1}{x^2} \\
f'''(x) &= 2x^{-3} = \frac{2}{x^3}
\end{aligned}$$

◁

9.5. Problems

Problem 9.1: Plot the following function:

$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto |x - n| \text{ for } n - \frac{1}{2} \le x < n + \frac{1}{2}, n \in \mathbb{Z} \end{cases}$$

- Is this function continuous?
- Is this function differentiable?

Problem 9.2: Evaluate the following derivatives.

1. $\frac{\mathrm{d}}{\mathrm{d}x}(-2x^2)$
2. $\frac{\mathrm{d}}{\mathrm{d}u}3e^u$
3. $\frac{\mathrm{d}}{\mathrm{d}x}(x^5 + \sqrt{2}x^3)$
4. $\frac{\mathrm{d}}{\mathrm{d}x}(x^3 - x^2 + 1)$
5. $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x + \cos y)$
6. $\frac{\mathrm{d}}{\mathrm{d}y}(\sin x + \cos y)$

Problem 9.3: Evaluate the following derivatives.

1. $\frac{\mathrm{d}}{\mathrm{d}x}(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + 1)$

2. $\frac{\mathrm{d}}{\mathrm{d}x}(\frac{1}{6}x^9 + \frac{2}{9}x^3 + \frac{1}{5}x^4)$
3. $\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + 2xy + y^2)$
4. $\frac{\mathrm{d}}{\mathrm{d}y}(x^2 - 2xy + y^2)$
5. $\frac{\mathrm{d}}{\mathrm{d}z}(x^2 - y^2)$
6. $\frac{\mathrm{d}}{\mathrm{d}x}x^1y^2z^3$
7. $\frac{\mathrm{d}}{\mathrm{d}y}x^1y^2z^3$
8. $\frac{\mathrm{d}}{\mathrm{d}z}x^1y^2z^3$

Problem 9.4: Evaluate the following derivatives.

1. $\frac{\mathrm{d}}{\mathrm{d}x}(x^{-2} + x^{-3})$
2. $\frac{\mathrm{d}}{\mathrm{d}x}(\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3})$
3. $\frac{\mathrm{d}}{\mathrm{d}x}(3x^{\frac{2}{3}} - 6x^{-\frac{1}{3}})$
4. $\frac{\mathrm{d}}{\mathrm{d}x}(\frac{1}{2}x^2 - \frac{1}{3}x^{-3})$
5. $\frac{\mathrm{d}}{\mathrm{d}x}(x^{\frac{1}{2}} - x^{-\frac{1}{2}})$
6. $\frac{\mathrm{d}}{\mathrm{d}x}(x^2 - 1)(x^{-2} + 1)$

Problem 9.5: Evaluate the following derivatives.

1. $\frac{\mathrm{d}}{\mathrm{d}x}e^{\mathrm{j}x}\sin x$
2. $\frac{\mathrm{d}}{\mathrm{d}x}\tan x \cos x$
3. $\frac{\mathrm{d}}{\mathrm{d}x}2x^{\frac{1}{2}}\sin x$
4. $\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^3 - x^2 + 1}{x^2 + 1}$
5. $\frac{\mathrm{d}}{\mathrm{d}x}\dfrac{x^2}{e^x}$
6. $\frac{\mathrm{d}}{\mathrm{d}x}\dfrac{\sin x}{2^x}$

Problem 9.6: Evaluate the following derivatives.

1. $\frac{\mathrm{d}}{\mathrm{d}x}e^{x^2}$
2. $\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}\sin(2\pi ft)$
3. $\frac{\mathrm{d}}{\mathrm{d}t}\hat{i}e^{\mathrm{j}(2\pi ft + \varphi)}$
4. $\frac{\mathrm{d}}{\mathrm{d}x}\sin^2(x)$
5. $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{\ln|x^2|}$
6. $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{\sin(3^x)}$

Problem 9.7: Evaluate the following derivatives ($n \in \mathbb{N}$).

1. $\frac{\mathrm{d}^2}{\mathrm{d}x^2}x^4$
2. $\frac{\mathrm{d}^3}{\mathrm{d}x^3}3^x$
3. $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\pi^x$
4. $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\hat{u}\sin(2\pi ft)$
5. $\frac{\mathrm{d}^4}{\mathrm{d}x^4}\sin(2x)$
6. $\frac{\mathrm{d}^n}{\mathrm{d}t^n}\hat{u}e^{\mathrm{j}(2\pi ft + \varphi)}$

10. Polynomials and rational functions

10.1. Polynomial function

Definition 10.1 (Polynomial function)**.** Let $n \in \mathbb{N} \cup \{0\}$, $a_k \in \mathbb{C}$, $k = 0, 1, \ldots, n$, $a_n \neq 0$ and $x \in \mathbb{C}$. We then call

$$\begin{aligned} p &= \sum_{k=0}^{n} a_k x^k \\ &= a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \end{aligned}$$

a *polynomial function* or just a *polynomial*. We say p is of n^{th} order and call a_k the *coefficients* of p. ◁

Remark: For our purposes we limit ourselves to real coefficients, i.e. $a_k \in \mathbb{R}$.

Example 10.1.

- $p_1(x) = x^2 - x + 1$ *is* a 2^{nd} order polynomial.
- $p_2(x) = x^5 - 1$ *is* a 5^{th} order polynomial.
- $p_3(x) = x^2 + 3/x + 1$ *is not* a polynomial due to the negative exponent of x in the second summand.
- $p_4(x) = x^3 - 2x^{1/2} + 3$ *is not* a polynomial since x of the second summand has a non-integer exponent.
- $p_5(x) = 2^{1/2} x^2 + x - 3$ *is* a second order polynomial ($2^{1/2} = \sqrt{2} \in \mathbb{R}$ is a coefficient for x^2).
- $p_6(x) = 1$ *is* a zero order polynomial.

◁

Theorem 10.2 (Properties of polynomials)**.**

- Polynomials are *continuous.*
- Polynomials are *differentiable.*
- *Sums* of polynomials are polynomials.

$$\sum_{k=0}^{n} a_k x^k + \sum_{k=0}^{m} b_k x^k = \sum_{k=0}^{\max(n,m)} c_k x^k$$

- *Multiples* of a polynomials are polynomials.

$$c \sum_{k=0}^{n} a_k x^k = \sum_{k=0}^{n} c\, a_k x^k$$

- *Products* of polynomials are polynomials.

$$\left(\sum_{k=0}^{n} a_k x^k\right) \cdot \left(\sum_{k=0}^{m} b_k x^k\right) = \sum_{k=0}^{n+m} c_k x^k$$

- *Compositions* of polynomials are polynomials.

$$\sum_{k=0}^{n} a_k \left(\sum_{l=0}^{m} b_l x^l\right)^k = \sum_{k=0}^{n \cdot m} c_k x^k$$

- *Derivatives* of non-constant polynomials are polynomials.

$$\frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=0}^{n} a_k x^k = \sum_{k=0}^{n-1} c_k x^k$$

◁

Definition 10.3 (Horner's method)**.** For a polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

we define Horner's method by

$$p(x) = ((\ldots(a_n x + a_{n-1})x + \ldots)x + a_1)x + a_0$$

◁

Example 10.2. We want to evaluate the polynomial

$$p(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$$

at $x = x_0$ only with addition and multiplication. In the standard form we need 4 additions and 10 multiplications.

Applying Horner's method

$$p(x_0) = (((a_4 x_0 + a_3)x_0 + a_2)x_0 + a_1)x_0 + a_0$$

we need again 4 additions, but only 4 multiplications.

Let's assume an n^{th} order polynomial with $a_k \neq 0$ for all $k = 0, \ldots, n$. In the standard notation we need n additions and $\frac{n(n+1)}{2}$ multiplications. Applying Horner's method we need again n additions, but only n multiplications. Hence, Horner's method requires a factor of $\frac{n+1}{2}$ less multiplications. ◁

10.2. Zeros

Theorem 10.4 (Number of zeros)**.** A polynomial $p : \mathbb{R} \to \mathbb{R}$ of order n has at most n zeros. ◁

Example 10.3. Let $x \in \mathbb{R}$.

- Zeros of $x^2 - 1$: $\{-1, 1\}$
- Zeros of $x^2 + 1$: $\{\}$
- Zeros of $x^3 - x$: $\{-1, 0, 1\}$
- Zeros of $x^3 + x$: $\{0\}$

◁

Theorem 10.5 (Zeros of odd order polynomials)**.** An odd order polynomial $p : \mathbb{R} \to \mathbb{R}$ has at least one zero. ◁

Theorem 10.6 (Separating zeros)**.** An n^{th} order polynomial p_n with a zero x_0 may be expressed by

$$p_n(x) = (x - x_0) \cdot p_{n-1}(x)$$

where $p_{n-1}(x)$ is a polynomial of order $n - 1$ with

$$p_{n-1}(x) = \frac{p_n(x)}{x - x_0}$$

◁

Example 10.4. The polynomial

$$p_3 : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^3 - 3x^2 - x + 3 \end{cases}$$

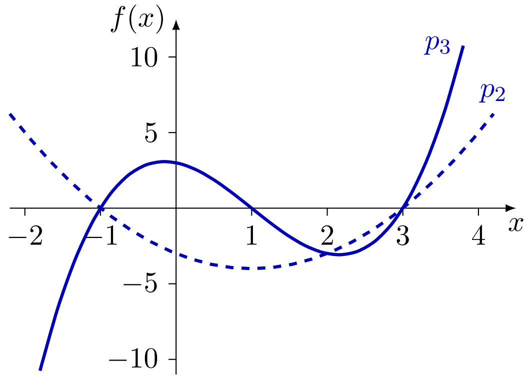
has (among other zeros) a zero at $x_0 = 1$. We divide the polynomial by this zero:

$$\begin{array}{llll}
(x^3 & -3x^2 & -x & +3)/(x-1) = x^2 - 2x - 3 \\
\underline{-(x^3} & \underline{-x^2)} & & \\
0 & -2x^2 & -x & \\
 & \underline{-(-2x^2} & \underline{+2x)} & \\
 & 0 & -3x & +3 \\
 & & \underline{-(\,-3x} & \underline{+3)} \\
 & & & 0
\end{array}$$

Hence we get

$$\begin{aligned} p_3(x) &= (x - x_0) \cdot p_2(x) \\ x^3 - 3x^2 - x + 3 &= (x - 1) \cdot (x^2 - 2x - 3) \end{aligned}$$

with $p_2(x)$ having two other zeros at $\{-1, 3\}$.



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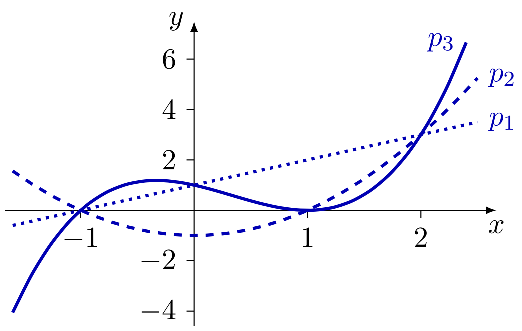
Definition 10.7 (Multiple zero)**.** If a polynomial $p : \mathbb{R} \to \mathbb{R}$ can be divided by a zero more than once without remainder we call it a *multiple zero*. ◁

Example 10.5. The polynomial

$$p_3 : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^3 - x^2 - x + 1 \end{cases}$$

has a double zero at 1. I.e.

$$\begin{aligned} p_3(x) &= x^3 - x^2 - x + 1 \\ p_2(x) &= \frac{p_3(x)}{x - 1} = x^2 - 1 \\ p_1(x) &= \frac{p_2(x)}{x - 1} = x + 1 \end{aligned}$$



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Theorem 10.8 (Fundamental theorem of algebra)**.** Every non-constant polynomial over complex numbers has at least one zero. ◁

Remark: The fundamental theorem of algebra includes complex zeros. Hence, polynomials like $p(x) = x^2 + 1$ have zeros in the complex domain.

If we divide a polynomial of order n by its zero $(x - x_0)$ we get a polynomial of order $n-1$. If this new polynomial is of order greater 0 it again will have a zero. Hence, the fundamental theorem of algebra indirectly states that any polynomial of order n has exactly n zeros.

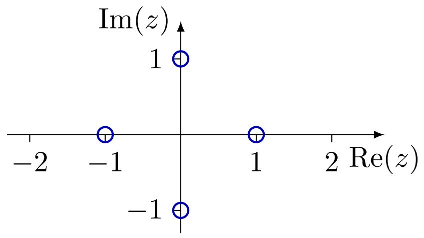
Example 10.6. The polynomial

$$f : \begin{cases} \mathbb{C} \to \mathbb{C} \\ z \mapsto z^4 - 1 \end{cases}$$

has four zeros:

$$\begin{aligned} f(z) = z^4 - 1 &= (z^2 - 1)(z^2 + 1) \\ &= (z - 1)(z + 1)(z - \mathrm{j})(z + \mathrm{j}) \end{aligned}$$

i.e. $z_1 = 1$, $z_2 = -1$, $z_3 = \mathrm{j}$ and $z_4 = -\mathrm{j}$.



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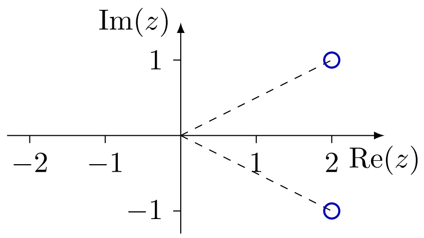
Theorem 10.9 (Conjugate complex zeros)**.** The complex zeros of a polynomial $p : \mathbb{C} \to \mathbb{C}$ with real coefficients appear in pairs of complex conjugates. I.e. for every complex zero z_1 with $\mathrm{Im}(z_1) \neq 0$ there exist another zero $z_2 = \overline{z_1}$. ◁

Example 10.7. The polynomial

$$f : \begin{cases} \mathbb{C} \to \mathbb{C} \\ z \mapsto z^2 - 4z + 5 \end{cases}$$

has two zeros at $z_{1,2} = 2 \pm \mathrm{j}$:

$$\begin{aligned} p_2(z) &= z^2 - 4z + 5 \\ p_1(z) &= \frac{p_2(z)}{z - z_1} = \frac{z^2 - 4z + 5}{z - (2 + \mathrm{j})} = z - 2 + \mathrm{j} \\ p_0 &= \frac{p_1(z)}{z - z_2} = \frac{z - 2 + \mathrm{j}}{z - (2 - \mathrm{j})} = 1 \end{aligned}$$



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Remark: When separating a polynomial into its zeros, the complex conjugate zeros may be kept together resulting in real coefficients. I.e. with complex conjugate zeros at $z_{1,2} = a \pm \mathrm{j}b$ we have

$$\begin{aligned} (z - z_1)(z - z_2) &= (z - (a + \mathrm{j}b))(z - (a - \mathrm{j}b)) \\ &= z^2 - 2az + a^2 + b^2 \\ &= z^2 + Az + B \end{aligned}$$

with

$$\begin{aligned} A &= -2a & B &= a^2 + b^2 \\ a &= -\frac{A}{2} & b &= \pm\sqrt{B - \frac{A^2}{4}} \end{aligned}$$

Theorem 10.10 (Rational root theorem)**.** Let

$$p_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

be an n^{th} order polynomial with integer coefficients $a_k \in \mathbb{Z}$, $k = 0, 1, \ldots, n$ and with $a_n \neq 0$ and $a_0 \neq 0$.

For each rational root $x \in \mathbb{Q}$ expressed as p/q, $p \in \mathbb{Z}$, $q \in \mathbb{N}$ with p and q having no common divisor we have:

- p is a divisor of a_0
- q is a divisor of a_n

◁

Remark: When searching for zeros of an higher order polynomial this theorem limits the number of candidates, see next example.

Example 10.8. We want to find the first zero of the polynomial

$$p_3(x) = 2x^3 + 3x^2 + 8x - 5$$

With p being an integer divisor of $a_0 = -5$, i.e. $p \in \{-5, -1, 1, 5\}$, and q being a natural divisor of $a_3 = 2$, i.e. $q \in \{1, 2\}$, we get

$$x_0 = \tfrac{p}{q} \in \{-5, -\tfrac{5}{2}, -1, -\tfrac{1}{2}, \tfrac{1}{2}, 1, \tfrac{5}{2}, 5\}$$

Trying these candidates results in $p_3(\frac{1}{2}) = 0$, i.e. $\frac{1}{2}$ is a rational root of $p_3(x)$.

If none of the candidates results in $p_3(x) = 0$ the polynomial has no rational zeros. However, due to the fundamental theorem of algebra the polynomial then has irrational and/or complex zeros. ◁

Remark: The term *root* is used here as a synonym to *zero of polynomial.*

For polynomials with $a_n = 1$ the rational roots can be integers only which further reduces the number of candidates.

10.3. Rational function

Definition 10.11 (Rational function)**.** Let p and q be polynomials and $q \neq 0$. We call

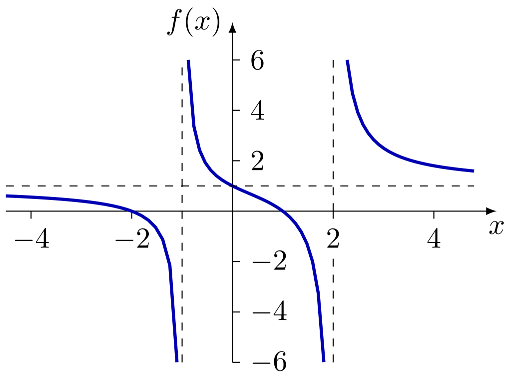
$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \frac{p(x)}{q(x)} \end{cases}$$

a *rational function*. ◁

Example 10.9. Let

$$f : \begin{cases} \mathbb{R} \setminus \{-1, 2\} \to \mathbb{R} \\ x \mapsto \dfrac{x^2 + x - 2}{x^2 - x - 2} \end{cases}$$

be a rational function. Since the denominator has two zeros at -1 and 2 we exclude these values from the domain.



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10.4. Properties of rational functions

Theorem 10.12 (Zeros of rational functions)**.** A rational function has a zero where the numerator has a zero and the denominator is not zero. ◁

Definition 10.13 (Pole)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a rational function. We call a point x_0 with the property

$$\lim_{x \to x_0} |f(x)| = \infty$$

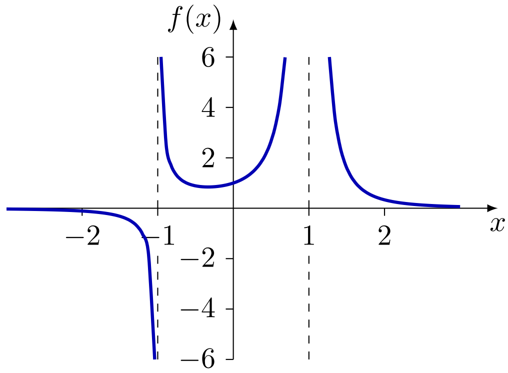
a *pole*. ◁

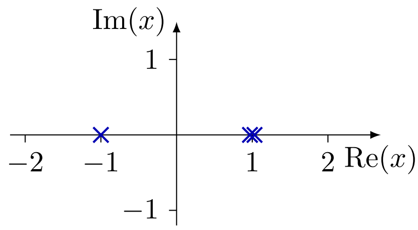
Remark: When taking the left- and right-side limit there are poles with equal and different signs for infinity.

Example 10.10. The rational function

$$f : \begin{cases} \mathbb{R} \setminus \{-1, 1\} \to \mathbb{R} \\ x \mapsto \dfrac{1}{x^3 - x^2 - x + 1} \end{cases}$$

has a pole at -1 with changing signs and a second pole at 1 with equal signs.





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Theorem 10.14 (Pole of a rational function)**.** A rational function has a pole at x_0 where the denominator has a zero and the numerator is not zero. ◁

Remark: What is the situation if both, numerator and denominator are zero? We need l'Hôpital's rule:

Theorem 10.15 (l'Hôpital's rule)**.** Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions, $x_0 \in \mathbb{R}$ and let the limit $\lim\limits_{x \to x_0} \dfrac{f'(x)}{g'(x)}$ exist (including $\pm\infty$). If either

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$$

or

$$\lim_{x \to x_0} |f(x)| = \lim_{x \to x_0} |g(x)| = \infty$$

then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

◁

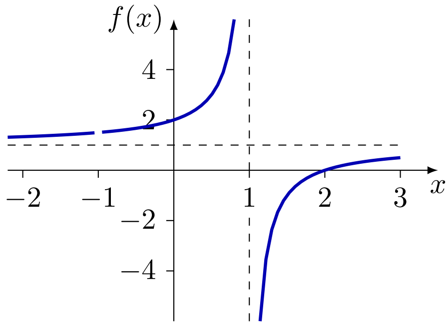
Example 10.11. The rational function

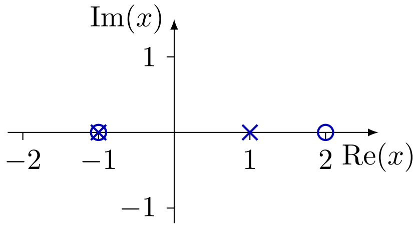
$$f : \begin{cases} \mathbb{R} \setminus \{-1, 1\} \to \mathbb{R} \\ x \mapsto \dfrac{x^2 - x - 2}{x^2 - 1} \end{cases}$$

has in its numerator zeros at -1 and 2 and in its denominator at -1 and 1. Hence, f has a zero at 2 and a pole at 1. For the common zero of numerator and denominator we apply l'Hôpital's rule:

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{2x - 1}{2x} = \frac{2(-1) - 1}{2(-1)} = \frac{3}{2}$$

I.e. for $x_0 = -1$ the rational function f approaches $\frac{3}{2}$, however, since -1 does not belong to the domain f will never have this value.





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Remark: A rational function is discontinuous at points where the denominator is zero. However, if the left- and right-side limits exist and are equal we call it a *removable discontinuity*.

Example 10.12. We want to prove the following theorem:

$$\lim_{n \to \infty} \sqrt[n]{n} = 1$$

Here is the proof applying l'Hôpital's rule in the third row:

$$\begin{aligned} \lim_{n \to \infty} \sqrt[n]{n} = \lim_{n \to \infty} n^{1/n} &= \lim_{n \to \infty} \exp\left(\ln(n^{1/n})\right) \\ &= \lim_{n \to \infty} \exp\left(\frac{1}{n} \ln(n)\right) \\ &= \exp\left(\lim_{n \to \infty} \frac{\ln(n)}{n}\right) = \exp\left(\lim_{n \to \infty} \frac{\frac{1}{n}}{1}\right) \\ &= \exp\left(\lim_{n \to \infty} \frac{1}{n}\right) = \exp(0) = 1 \end{aligned}$$

◁

10.5. Partial fraction decomposition

Rational functions with large order polynomials in numerator and denominator are difficult to handle as they are. We are looking for a technique to split them into a sum of more basic fractions.

The technique of *partial fraction decomposition* changes a high-order rational function into a sum of low-order rational functions.

We need this technique later e.g. to perform integration or Laplace-transform of high-order rational functions.

10.5.1. Order of numerator and denominator

In a first step we need to assure that the order of the numerator polynomial is less than the order of the denominator polynomial.

In German we call this "*echt gebrochenrationale Funktion*" where the term *echt* (Engl. *true*) indicates the higher order in the denominator. In English we do not have a special term for this.

If the order of the denominator polynomial is less or equal than the one of the numerator polynomial, we perform a polynomial division where the remainder is a fraction with the required order of the polynomials.

Example 10.13. For the rational function

$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \dfrac{x^3 + 3x^2 - x - 3}{x^2 - 4} \end{cases}$$

the order of the numerator polynomial is not less than the order of the denominator polynomial. Hence we perform a polynomial division:

$$\begin{array}{l} (x^3 + 3x^2 - x - 3)/(x^2 - 4) = x + 3 + \frac{3x+9}{x^2-4} \\ \underline{-(x^3 \qquad\quad -4x)} \\ \qquad +3x^2 + 3x \quad -3 \\ \quad \underline{-(+3x^2 \qquad\quad -12)} \\ \qquad\qquad +3x \quad +9 \end{array}$$

Hence we have:

$$f(x) = \frac{x^3 + 3x^2 - x - 3}{x^2 - 4} = x + 3 + \frac{3x + 9}{x^2 - 4}$$

◁

To further split the remaining rational function into summands we apply the following techniques:

10.5.2. Partial fractions for real zeros of denominator

Let us assume a rational function f with the order n of the polynomial of its denominator being greater than the one of its numerator.

If the polynomial of the denominator has real zeros only we create the partial fractions as follows:

- for a single (not multiple) zero x_0:

$$\frac{A}{x-x_0}$$

- for a double zero x_0:

$$\frac{A_1}{x-x_0}+\frac{A_2}{(x-x_0)^2}$$

- for an n-multiple zero x_0:

$$\frac{A_1}{x-x_0}+\frac{A_2}{(x-x_0)^2}+\ldots+\frac{A_n}{(x-x_0)^n}$$

We set the sum of these fractions equal to f and multiply by the denominator of f. The equation must be true for any x. If we set x to the zeros of the denominator of f we get simple equations that reveal the unknown constants.

Example 10.14. We take the remainder of the previous example:

$$f:\begin{cases}\mathbb{R}\to\mathbb{R}\\ x\mapsto \dfrac{3x+9}{x^2-4}\end{cases}$$

We have two real zeros for the denominator $x_1=2$ and $x_2=-2$ and get:

$$f(x)=\frac{3x+9}{x^2-4}=\frac{A}{x-2}+\frac{B}{x+2}$$

We multiply with the denominator of $f(x)$:

$$3x+9=\frac{A(x^2-4)}{x-2}+\frac{B(x^2-4)}{x+2}$$

Since the denominator zeros of the partial fractions are also denominator zeros of f, it is always possible to cancel them down:

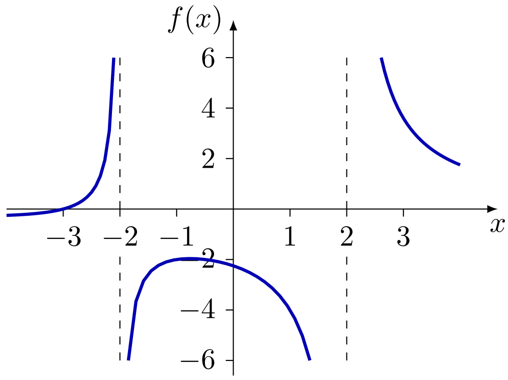
$$3x+9=A(x+2)+B(x-2)$$

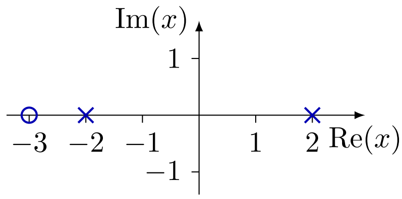
We now set x to 2 and -2 to find the constants A and B:

$$\begin{aligned} x=2: \quad 3\cdot 2+9 &= A(2+2)+B(2-2)\\ 15&=4A\\ A&=\frac{15}{4}\\ x=-2: \quad 3(-2)+9&=A(-2+2)+B(-2-2)\\ 3&=-4B\\ B&=-\frac{3}{4}\end{aligned}$$

Hence we get:

$$f(x)=\frac{3x+9}{x^2-4}=\frac{\frac{15}{4}}{x-2}-\frac{\frac{3}{4}}{x+2}$$





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Example 10.15. The rational function

$$f:\begin{cases}\mathbb{R}\to\mathbb{R}\\ x\mapsto \dfrac{x^2+2x+1}{x^3-3x+2}\end{cases}$$

has in its denominator a single zero at -2 and a double zero at 1. Hence we separate f into three partial fractions and multiply by its denominator:

$$\begin{aligned}\frac{x^2+2x+1}{x^3-3x+2}&=\frac{A}{x+2}+\frac{B_1}{x-1}+\frac{B_2}{(x-1)^2}\\ x^2+2x+1&=(x-1)^2A+(x-1)(x+2)B_1\\ &\quad+(x+2)B_2\end{aligned}$$

Setting x to -2 and 1 gives:

$$x=-2: \quad 1=(-3)^2A \quad\Rightarrow\quad A=\frac{1}{9}$$

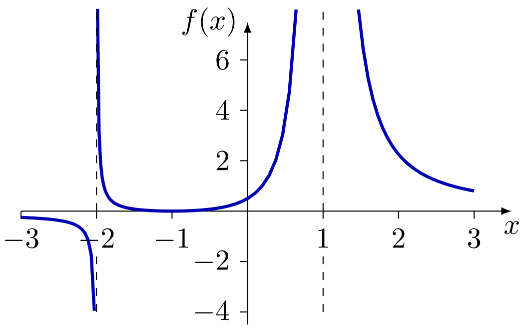
$$x = 1: \qquad 4 = 3B_2 \qquad \Rightarrow \qquad B_2 = \frac{4}{3}$$

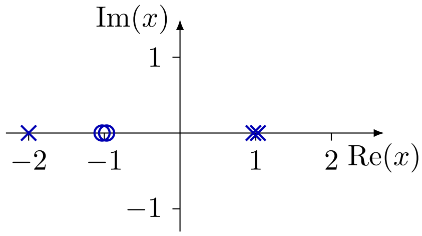
To find B_1 we set x to any value other than the zeros, e.g. $x=0$:

$$x = 0: \quad 1 = \frac{1}{9} - 2B_1 + \frac{8}{3} \quad \Rightarrow \quad B_1 = \frac{8}{9}$$

Hence, we get:

$$f(x) = \frac{x^2+2x+1}{x^3-3x+2} = \frac{\frac{1}{9}}{x+2} + \frac{\frac{8}{9}}{x-1} + \frac{\frac{4}{3}}{(x-1)^2}$$





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10.5.3. Partial fractions for complex zeros of denominator

Let f be a rational function with the order of its denominator-polynomial being greater than the order of its numerator-polynomial.

This time we assume that the denominator polynomial also includes complex zeros. For real coefficients every complex zero z_1 results in a second complex conjugate zero $\overline{z_1}$. We combine the pairs of complex conjugate zeros:

$$\begin{aligned}(x-z_1)(x-\overline{z_1}) &= x^2 - (z_1+\overline{z_1})x + z_1\overline{z_1} \\ &= x^2 + ax + b\end{aligned}$$

with

$$a = -(z_1 + \overline{z_1}) \qquad \text{and} \qquad b = z_1\overline{z_1}$$

We use this term as denominator for partial fractions:

- for a single (not multiple) pair of zeros:
$$\frac{Ax+B}{x^2+ax+b}$$
- for a double pair of zeros:
$$\frac{A_1x+B_1}{x^2+ax+b} + \frac{A_2x+B_2}{(x^2+ax+b)^2}$$
- for an n-multiple pair of zeros:
$$\frac{A_1x+B_1}{x^2+ax+b} + \ldots + \frac{A_nx+B_n}{(x^2+ax+b)^n}$$

Example 10.16. The denominator of the rational function

$$f: \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \dfrac{x^2-4}{x^3-x^2+2} \end{cases}$$

has a real zero at -1 and a pair of complex conjugate zeros at $1 \pm \mathrm{j}$. Hence we separate f into two partial fractions:

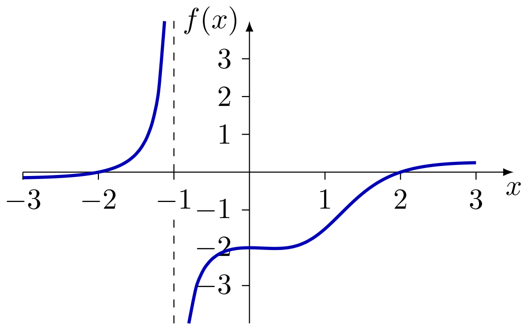
$$\begin{aligned}\frac{x^2-4}{x^3-x^2+2} &= \frac{A}{x+1} + \frac{Bx+C}{x^2-2x+2} \\ x^2-4 &= (x^2-2x+2)A + (x+1)(Bx+C)\end{aligned}$$

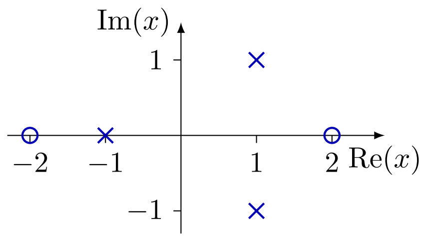
We set x to -1, 0 and 1:

$$\begin{aligned}
x=-1: &\quad -3 = 5A && \Rightarrow \quad A = -\frac{3}{5} \\
x=0: &\quad -4 = -\frac{6}{5} + C && \Rightarrow \quad C = -\frac{14}{5} \\
x=1: &\quad -3 = -\frac{3}{5} + 2B - \frac{28}{5} && \Rightarrow \quad B = \frac{8}{5}
\end{aligned}$$

Hence, we get:

$$f(x) = \frac{x^2-4}{x^3-x^2+2} = \frac{-\frac{3}{5}}{x+1} + \frac{\frac{8}{5}x - \frac{14}{5}}{x^2-2x+2}$$





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10.6. Taylor polynomial

Polynomials are readily to handle e.g. for differentiation or, as we see later, for integration. Hence, if it is possible to approximate functions by polynomials, further calculations become easy.

If we want to approximate a function f at point x_0 by a zero order polynomial (i.e. a constant) we simply take the function value $f(x_0)$ as the coefficient a_0 of the polynomial p_0:

$$p_0 = a_0 = f(x_0)$$

To improve the approximation we may increase the order to a first order polynomial and adjust its slope to the slope of $f(x)$ at x_0. I.e. the first derivative of function and polynomial shall be equal and we get the equations:

$$\begin{aligned} p_1(x_0) &= f(x_0) \\ p_1'(x_0) &= f'(x_0) \end{aligned}$$

In order to further improve the approximation we may include higher order derivatives:

$$\begin{aligned} p_n(x_0) &= f(x_0) \\ p_n'(x_0) &= f'(x_0) \\ p_n''(x_0) &= f''(x_0) \\ \vdots \quad & \quad \vdots \\ p_n^{(n)}(x_0) &= f^{(n)}(x_0) \end{aligned}$$

The resulting polynomial is the *Taylor polynomial*:

Definition 10.16 (Taylor polynomial)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be an n times differentiable function. We define the *Taylor polynomial* as

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

with $x_0 \in \mathbb{R}$ as the *expansion point*. $\triangleleft$

Example 10.17. We want to approximate the function

$$f : \begin{cases} \mathbb{R}_{>-1} \to \mathbb{R} \\ x \mapsto \frac{1}{x+1} \end{cases}$$

by Taylor polynomials at expansion point $x_0 = 0$. The derivatives of f are:

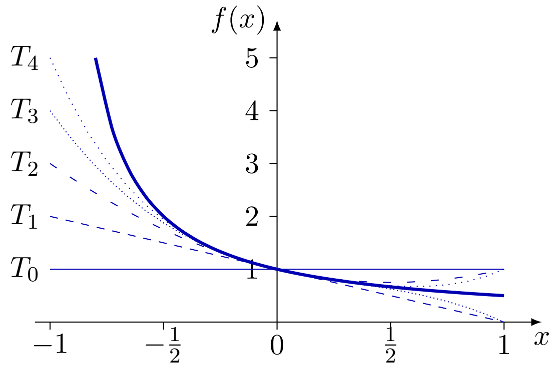
$$\begin{aligned} f'(x) &= \frac{-1}{(x+1)^2} \\ f''(x) &= \frac{2}{(x+1)^3} \\ f'''(x) &= \frac{-6}{(x+1)^4} \\ \vdots \quad & \quad \vdots \\ f^{(n)}(x) &= \frac{(-1)^n n!}{(x+1)^{n+1}} \end{aligned}$$

With $x_0 = 0$ we get

$$\begin{aligned} T_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \\ &= \sum_{k=0}^{n} \frac{(-1)^k}{(x_0+1)^{k+1}}(x - x_0)^k \\ &= \sum_{k=0}^{n} (-x)^k \end{aligned}$$

I.e.

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= 1 - x \\ T_2(x) &= 1 - x + x^2 \\ T_3(x) &= 1 - x + x^2 - x^3 \\ T_4(x) &= 1 - x + x^2 - x^3 + x^4 \\ \vdots \quad & \quad \vdots \end{aligned}$$



$\triangleleft$

An application of the Taylor polynomial is the *propagation of uncertainty*:

10.7. Propagation of uncertainty

In engineering we perform measurements with an uncertainty due to e.g. tolerances of the equipment. For an measured value x we may express the uncertainty either absolutely or relatively:

$$\text{absolute uncertainty: } \Delta x$$

$$\text{relative uncertainty: } \frac{\Delta x}{|x|}$$

Example 10.18.

For a resistor with colour code brown-black-orange-gold we have:

$$\begin{aligned} R &= 10\,\mathrm{k}\Omega \quad \text{(brown-black-orange)} \\ \tfrac{\Delta R}{R} &= 5\,\% \qquad \text{(gold)} \\ \Delta R &= R \cdot \tfrac{\Delta R}{R} = 0.5\,\mathrm{k}\Omega \end{aligned}$$

◁

Often an investigated value is a function $f : \mathbb{R}^n \to \mathbb{R}$ of some input values x_k, $k = 1, 2, \ldots, n$ each having an uncertainty Δx_k. Applying a first order Taylor polynomial w.r.t. one input value x_k the constant p_0 of the polynomial T_1 gives the expected value and the first order coefficient p_1 the factor for propagation of uncertainty Δx_k:

$$T_1 \approx p_0 \pm p_1 \cdot \Delta x_k$$

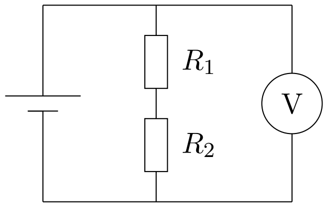
I.e. multiplying p_1 of the Taylor polynomial by the uncertainty of the input value Δx_k gives the propagated uncertainty of the input value x_k to the output value f. We do this for all arguments and sum up the effect to the output value:

Theorem 10.17 (Propagation of uncertainty)**.** Let $f(x_1, x_2, \ldots, x_n)$ be a function $\mathbb{R}^n \to \mathbb{R}$ with n arguments $x_1, x_2, \ldots, x_n$ having uncertainties $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$, respectively. For small uncertainties of the arguments the uncertainty of the function Δf approaches:

$$\Delta f \approx \sum_{k=1}^{n} \left| \frac{\partial f}{\partial x_k} \right| \Delta x_k$$

◁

Example 10.19. We connect a battery to a series of two resistors and evaluate the absorbed power. The applied voltage is $9.6\,\mathrm{V}$ measured with a volt meter having $1\,\%$ tolerance. The two resistors have a resistance of 100 and $220\,\Omega$ each with an uncertainty of $5\,\%$.



What is the expected absorbed power P and how do the uncertainties propagate to the absorbed power ΔP? From electrical engineering we know the expected absorbed power to be

$$P = \frac{U^2}{R_1 + R_2} = \frac{(9.6\,\mathrm{V})^2}{100\,\Omega + 220\,\Omega} = 288\,\mathrm{mW}$$

For ΔP we apply the theorem of propagation of uncertainties:

$$\begin{aligned} \Delta P &\approx \sum_{k=1}^{3} \left| \frac{\partial P}{\partial x_k} \right| \Delta x_k \\ &= \left| \frac{\partial P}{\partial U} \right| \Delta U + \left| \frac{\partial P}{\partial R_1} \right| \Delta R_1 + \left| \frac{\partial P}{\partial R_2} \right| \Delta R_2 \\ &= \left| \frac{2U}{R_1 + R_2} \right| \Delta U + \left| \frac{-U^2}{(R_1 + R_2)^2} \right| \Delta R_1 \\ &\quad + \left| \frac{-U^2}{(R_1 + R_2)^2} \right| \Delta R_2 \\ &= 5.76\,\mathrm{mW} + 4.5\,\mathrm{mW} + 9.9\,\mathrm{mW} \\ &= 20.16\,\mathrm{mW} \end{aligned}$$

The relative uncertainty of P is:

$$\frac{\Delta P}{P} \approx \frac{20.16\,\mathrm{mW}}{288\,\mathrm{mW}} = 0.07 = 7\,\%$$

This calculation may be simplified by the following corollaries: ◁

Corollary 10.18 (Uncertainty of sums and differences)**.** Let the numbers $x_1, x_2 \in \mathbb{R}$ have the absolute uncertainties $\Delta x_1, \Delta x_2$, respectively. Then the sum $x_1 + x_2$ and the difference $x_1 - x_2$ have the absolute uncertainty $\Delta x_1 + \Delta x_2$.

I.e. for a sum or difference the *absolute* uncertainties sum up. ◁

Corollary 10.19 (Uncertainty of products and quotients)**.** Let the numbers $x_1, x_2 \in \mathbb{R}$ have the relative uncertainties $\frac{\Delta x_1}{|x_1|}, \frac{\Delta x_2}{|x_2|}$, respectively. Then the product $x_1 \cdot x_2$ and the quotient x_1/x_2 (in this case $x_2 \neq 0$) have the uncertainty $\frac{\Delta x_1}{|x_1|} + \frac{\Delta x_2}{|x_2|}$.

I.e. for a product or quotient the *relative* uncertainties sum up. ◁

Corollary 10.20 (Uncertainty of powers)**.** Let the number $x \in \mathbb{R}$ have the relative uncertainty $\frac{\Delta x}{|x|}$ and let $y \in \mathbb{R}$. Then the power x^y has an uncertainty $\frac{\Delta x}{|x|} \cdot |y|$.

I.e. for powers the *relative* uncertainty of the base multiplies by the absolute of the exponent. ◁

Example 10.20. We come back to the previous example. What is the relative uncertainty of the absorbed power by applying the three corollaries? We investigate the relative uncertainties of numerator, denominator and fraction.

Numerator (third corollary):

$$\frac{\Delta(U^2)}{U^2} = \frac{\Delta U}{U} \cdot 2 = 1\,\% \cdot 2 = 2\,\%$$

Denominator (first corollary):

$$\begin{aligned}\frac{\Delta(R_1 + R_2)}{R_1 + R_2} &= \frac{\Delta R_1 + \Delta R_2}{R_1 + R_2} \\ &= \frac{R_1 \frac{\Delta R_1}{R_1} + R_2 \frac{\Delta R_2}{R_2}}{R_1 + R_2} \\ &= \frac{R_1 \cdot 5\,\% + R_2 \cdot 5\,\%}{R_1 + R_2} = 5\,\%\end{aligned}$$

Fraction (second corollary):

$$\frac{\Delta P}{P} = \frac{\Delta(U^2)}{U^2} + \frac{\Delta(R_1 + R_2)}{R_1 + R_2} = 2\,\% + 5\,\% = 7\,\%$$

I.e. for the numerator the uncertainty of the voltage doubles due to the power of two. For the denominator the relative uncertainties of the resistors do not change. Finally, for the fraction the relative uncertainties of numerator and denominator sum up. ◁

10.8. Problems

Problem 10.1: Which of the following functions $f : \mathbb{R} \to \mathbb{R}$ are polynomials? If so, what is the order?

1. $f(x) = 1 + x + x^2 + x^3 + x^4$
2. $f(x) = x^3 - 3$
3. $f(x) = 0$
4. $f(x) = \sqrt{3x^4} - x + 1$
5. $f(y) = \sqrt{x}$
6. $f(z) = z^{100}$

Problem 10.2: Let p_1 and p_2 be polynomials of 3rd and 4th order, respectively. Which of the following terms are polynomials? If so, what is the order?

1. $p_1 + p_2$
2. $p_1 \cdot p_2$
3. $p_1' + 2p_2$
4. $3p_1 + 2p_2'$
5. $p_1(p_2)$
6. $p_2(p_1^2)$

Problem 10.3: Perform polynomial division:

1. $(x^3 - 6x^2 + 11x - 6)/(x - 3)$
2. $(x^3 - 6x^2 + 11x - 6)/(x - 2)$
3. $(x^3 - 6x^2 + 11x - 6)/(x - 1)$
4. $(x^4 - 5x^2 + 4)/(x^2 - x - 2)$

Problem 10.4: Write the following polynomials as the product of their zeros.

1. $p_1(x) = x^2 - 1$
2. $p_2(x) = 2x^3 - 2x^2 - 4x$
3. $p_3(x) = x^2 - (a + b)x + ab$
4. $p_4(z) = z^2 - 2xz + x^2$
5. $p_5(x) = x^2 + 2x + 2$
6. $p_6(x) = x^3 - 3x^2 + 7x - 5$

Problem 10.5: For a polynomial with real coefficients a complex zero z_1 is known. What can you say about other zeros and what is the minimum order of the polynomial?

Problem 10.6: Plot the following function:

$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \dfrac{x^2 - 1}{x^2 - 4} \end{cases}$$

Problem 10.7: Analyse the following functions with respect to zeros and poles.

1. function of previous problem.
2. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \dfrac{1}{x^2 + x - 2} \end{cases}$

3. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \dfrac{x^2 - 1}{x^2 - 4x + 4} \end{cases}$

4. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \dfrac{x^2 - x - 2}{x^2 - 4} \end{cases}$

Problem 10.8: Create a rational function $f : \mathbb{R} \to \mathbb{R}$ with zeros at ± 1, poles at ± 3.

Problem 10.9: Separate from the following functions $f : \mathbb{R} \to \mathbb{R}$ a polynomial in order that the remaining rational function has a higher order in its denominator.

1. $f(x) = \dfrac{x^4}{x^2 - 1}$

2. $f(x) = \dfrac{x^4 - 2x^2 + 1}{x^2 + x - 2}$

3. $f(x) = \dfrac{x^6}{x^3 - x^2 + x - 1}$

4. $f(x) = \dfrac{x^5 - x^4 + x^3}{x^2 - x + 1}$

Problem 10.10: Perform partial fraction decomposition for the functions $f : \mathbb{R} \to \mathbb{R}$.

1. $f(x) = \dfrac{1}{x^2 - 1}$

2. $f(x) = \dfrac{2x + 3}{x^2 + 4x + 4}$

3. $f(x) = \dfrac{x^2 + 1}{x^3 + 2x^2 + 2x}$

4. $f(x) = \dfrac{x^3 + x}{x^4 + 2x^3 + x^2 - 2x - 2}$

Problem 10.11: Evaluate the 2^{nd} order Taylor polynomial at expansion point $x_0 = 0$ of the exponential function.

Problem 10.12: Evaluate the 1^{st} order Taylor polynomial at expansion point $x_0 = 1$ of the natural logarithm function.

11. System of linear equations

11.1. Introduction

Example 11.1. "If John gave 1 € to Judy they both would have the same amount of money. If Judy gave 1 € to John he would have twice as much money as her. How much money does each of them have?"

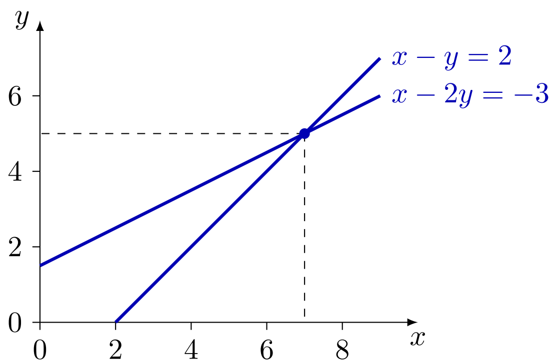
How do we find out? We first define x and y as John's and Judy's amount of money, respectively. Then we express the first two sentences as two equations:

$$x - 1 = y + 1$$
$$x + 1 = 2 \cdot (y - 1)$$

We multiply the brackets, bring all constant summands to the right and all summands including x or y to the left:

$$x - y = 2$$
$$x - 2y = -3$$

These two equations can be visualized as straight lines in a Cartesian diagram:



We subtract the first from the second equation and leave the first as it is:

$$x - y = 2$$
$$-y = -5$$

Then we subtract the second from the first equation and multiply the second by -1:

$$x = 7$$
$$y = 5$$

To check the result we insert the x and y into the equations we derived above.

$$7 - 1 = 5 + 1 \qquad \Rightarrow \text{ok}$$
$$7 + 1 = 2 \cdot (5 - 1) \qquad \Rightarrow \text{ok}$$

Hence, John has 7 € and Judy 5 €. ◁

This example deals with a small system of linear equations. With two equations and two unknowns we found a single solution. Is this always possible? Is there always a single solution?

A system of linear equations may deal with more than two unknowns. It may be used to numerically solve integrals or differential equations. In three dimensional field problems the number of unknowns may increase to several million or more!

In this chapter we get a first idea on systems of linear equations, derive a general approach to solve them and discuss whether they are solvable at all.

11.2. Definition

Definition 11.1 (System of linear equations)**.** We define a *system of linear equations* (*SLE*) as

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

with the coefficients $a_{11}, a_{12}, \dots, a_{mn} \in \mathbb{K}$, the constant terms $b_1, b_2, \dots, b_m \in \mathbb{K}$, $m \in \mathbb{N}$ as the number of equations and $n \in \mathbb{N}$ as the number of unknowns $x_1, \dots, x_n \in \mathbb{K}$.

If $b_1, \dots, b_m$ are all zero we call it a *homogeneous SLE* and an *inhomogeneous SLE* otherwise. ◁

Example 11.2. In the introductory example we derived an SLE with two equations containing the two unknowns x and y:

$$x - y = 2$$

$$x - 2y = -3$$

Hence we get

$$\begin{aligned} a_{11} &= 1, & a_{12} &= -1, & b_1 &= 2, \\ a_{21} &= 1, & a_{22} &= -2, & b_2 &= -3. \end{aligned}$$

◁

Remark: An SLE may be expressed by

$$\underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{x} \underbrace{=}_{=} \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}}_{b}$$

where A denotes the *coefficient matrix*, x a *solution vector* and b the *constant vector* or *inhomogeneous vector*.

Example 11.3. We rewrite the SLE of the introductory example

$$x_1 - x_2 = 2$$
$$x_1 - 2x_2 = -3$$

and get

$$\underbrace{\begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x} \underbrace{=}_{=} \underbrace{\begin{pmatrix} 2 \\ -3 \end{pmatrix}}_{b}$$

◁

Remark: As an expression for an SLE we may write in short

$$\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right) = (A \mid b)$$

which is the *extended coefficient matrix*.

Example 11.4. For the introductory example we get:

$$\left(\begin{array}{cc|c} 1 & -1 & 2 \\ 1 & -2 & -3 \end{array}\right) = (A \mid b)$$

◁

11.3. Gauss-Jordan elimination

11.3.1. Elementary row operations

To solve an SLE we apply three *elementary row operations*:

Theorem 11.2 (Row switching)**.** *If an equation of an SLE is switched with another one, the values of the unknowns do not change.* ◁

Theorem 11.3 (Row multiplication)**.** *An equation of an SLE may be multiplied by a constant $c \in \mathbb{K}$, $c \neq 0$, without changing the values of the unknowns.* ◁

Theorem 11.4 (Row addition)**.** *If an equation of an SLE is added to another equation of the same SLE, then the values of the unknowns do not change.* ◁

11.3.2. An example

Let's assume the following SLE:

$$\begin{array}{rcrcrcr} 2x_1 & - & 2x_2 & + & x_3 & = & -3 \\ -x_1 & + & 3x_2 & + & x_3 & = & 3 \\ x_1 & + & 3x_2 & - & 2x_3 & = & 7 \end{array}$$

we write it as extended coefficient matrix:

$$\left(\begin{array}{rrr|r} 2 & -2 & 1 & -3 \\ -1 & 3 & 1 & 3 \\ 1 & 3 & -2 & 7 \end{array}\right)$$

The aim is to convert the left part of the extended coefficient matrix into an identity matrix. We start with the first column and change the the top left element to one by multiplying row 1 by $\frac{1}{2}$:

$$\left(\begin{array}{rrr|r} 1 & -1 & \frac{1}{2} & -\frac{3}{2} \\ -1 & 3 & 1 & 3 \\ 1 & 3 & -2 & 7 \end{array}\right)$$

Then we add row 1 to row 2 and subtract row 1 from row 3:

$$\left(\begin{array}{rrr|r} 1 & -1 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 2 & \frac{3}{2} & \frac{3}{2} \\ 0 & 4 & -\frac{5}{2} & \frac{17}{2} \end{array}\right)$$

Now the first column is as expected and we proceed with the second column. We multiply row 2 by $\frac{1}{2}$:

$$\left(\begin{array}{rrr|r} 1 & -1 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & \frac{3}{4} & \frac{3}{4} \\ 0 & 4 & -\frac{5}{2} & \frac{17}{2} \end{array}\right)$$

Then we add row 2 to row 1 and subtract the quadruple of row 2 from row 3:

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{5}{4} & -\frac{3}{4} \\ 0 & 1 & \frac{3}{4} & \frac{3}{4} \\ 0 & 0 & -\frac{11}{2} & \frac{11}{2} \end{array}\right)$$

Now the first and second column is as expected and we focus on the third column. We multiply row 3 by $-\frac{2}{11}$:

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{5}{4} & -\frac{3}{4} \\ 0 & 1 & \frac{3}{4} & \frac{3}{4} \\ 0 & 0 & 1 & -1 \end{array}\right)$$

Then we subtract $\frac{5}{4}$ of row 3 from row 1 and subtract $\frac{3}{4}$ of row 3 from row 2:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & -1 \end{array}\right)$$

Since the left part is the identity matrix the right column gives us the values of the unknown:

$$\begin{array}{ccccccc} 1\cdot x_1 & + & 0\cdot x_2 & + & 0\cdot x_3 & = & \frac{1}{2} \\ 0\cdot x_1 & + & 1\cdot x_2 & + & 0\cdot x_3 & = & \frac{3}{2} \\ 0\cdot x_1 & + & 0\cdot x_2 & + & 1\cdot x_3 & = & -1 \end{array}$$

I.e. the solution of the SLE:

$$x_1 = \frac{1}{2} \qquad x_2 = \frac{3}{2} \qquad x_3 = -1$$

We check the result by inserting x_1, x_2 and x_3 into the original equations:

$$\begin{array}{rcl} 2\cdot\frac{1}{2} - 2\cdot\frac{3}{2} - 1 = -3 & & \Rightarrow \text{ ok} \\ -\frac{1}{2} + 3\cdot\frac{3}{2} - 1 = 3 & & \Rightarrow \text{ ok} \\ \frac{1}{2} + 3\cdot\frac{3}{2} + 2 = 7 & & \Rightarrow \text{ ok} \end{array}$$

In this example we were lucky to end with an identity matrix, i.e.

$$a_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{otherwise} \end{cases}$$

This is not always the case. To find a general approach we need the so called *reduced row echelon form* of a matrix:

11.3.3. Reduced row echelon form, rref

Definition 11.5 (Reduced row echelon form)**.** Let $M(m \times n, \mathbb{K})$ be a matrix with m rows and n columns of elements in $\mathbb{K}$. We say M has *reduced row echelon form (rref)* if

1. All rows containing zeros only (i.e. zero-rows) are at the end of M (i.e. the bottom of M).
2. The leading entry of non-zero rows (i.e. the first from the left unequal zero) is one.
3. From row 2 on the leading entry of non-zero rows is further to the right than all previous leading entries (i.e. the rows above).
4. All entities above and below a leading one are zero.

◁

Theorem 11.6 (Reduced row echelon form)**.** Any matrix $M(m \times n, \mathbb{K})$ can be converted into reduced row echelon form by elementary row operations. ◁

To change a matrix into its reduced row echelon form we proceed as in the previous example:

1. Start with row $k = 1$.
2. If there exists a row below row k with a leading entry further to the left than in row k switch the two rows.
3. Divide row k by its leading entry. (Now the leading entry of row k is one.)
4. Add multiples of row k to the other rows to change all elements above and below the leading entry of row k to zero.
5. Repeat steps two to four for $k = 2 \dots m$

Remark: To indicate the change of a matrix A into its reduced row echelon form we write in short:

$$\operatorname{rref}(A)$$

Example 11.5. We want to change the SLE

$$\begin{array}{ccccccccc} & & 2x_2 & + & 3x_3 & + & 4x_4 & = & 1 \\ x_1 & + & 4x_2 & + & 3x_3 & + & 5x_4 & = & 6 \\ x_1 & + & 4x_2 & + & 5x_3 & + & x_4 & = & 4 \\ x_1 & + & 3x_2 & + & 2x_3 & + & 2x_4 & = & 5 \end{array}$$

into reduced row echelon form. Writing it as extended coefficient matrix and indicating the row operations on the right we get:

$$\begin{array}{c} \left(\begin{array}{cccc|c} 0 & 2 & 3 & 4 & 1 \\ 1 & 4 & 3 & 5 & 6 \\ 1 & 4 & 5 & 1 & 4 \\ 1 & 3 & 2 & 2 & 5 \end{array}\right) \end{array} \begin{array}{l} = R_2 \\ = R_1 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 4 & 3 & 5 & 6 \\ 0 & 2 & 3 & 4 & 1 \\ 1 & 4 & 5 & 1 & 4 \\ 1 & 3 & 2 & 2 & 5 \end{array}\right) \begin{array}{l} \\ \\ -R_1 \\ -R_1 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 4 & 3 & 5 & 6 \\ 0 & 2 & 3 & 4 & 1 \\ 0 & 0 & 2 & -4 & -2 \\ 0 & -1 & -1 & -3 & -1 \end{array}\right) \begin{array}{l} \\ \cdot\frac{1}{2} \\ \\ \\ \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 4 & 3 & 5 & 6 \\ 0 & 1 & \frac{3}{2} & 2 & \frac{1}{2} \\ 0 & 0 & 2 & -4 & -2 \\ 0 & -1 & -1 & -3 & -1 \end{array}\right) \begin{array}{l} -4R_2 \\ \\ \\ +R_2 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -3 & -3 & 4 \\ 0 & 1 & \frac{3}{2} & 2 & \frac{1}{2} \\ 0 & 0 & 2 & -4 & -2 \\ 0 & 0 & \frac{1}{2} & -1 & -\frac{1}{2} \end{array}\right) \begin{array}{l} \\ \\ \cdot\frac{1}{2} \\ \\ \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -3 & -3 & 4 \\ 0 & 1 & \frac{3}{2} & 2 & \frac{1}{2} \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & \frac{1}{2} & -1 & -\frac{1}{2} \end{array}\right) \begin{array}{l} +3R_3 \\ -\frac{3}{2}R_3 \\ \\ -\frac{1}{2}R_3 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -9 & 1 \\ 0 & 1 & 0 & 5 & 2 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The last row became a zero-row. Column 4 cannot be further modified without destroying what we achieved for the first three columns. Recalling the definition we realize that the matrix is in reduced row echelon form. ◁

Rectangular coefficient matrices may also be converted into rref:

Example 11.6. We want to change the SLE

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 2 \\ x_2 + 2x_3 &= 1 \\ x_1 + x_3 &= 1 \\ 2x_2 + 3x_3 &= 2 \end{aligned}$$

into reduced row echelon form and write it as extended coefficient matrix. To indicate what we do we write the row operations to the right of the matrix.

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 3 & 2 \end{array}\right) \begin{array}{l} \\ \\ -R_1 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 2 & 3 & 2 \end{array}\right) \begin{array}{l} -R_2 \\ \\ +R_2 \\ -2R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right) \begin{array}{l} \\ -2R_3 \\ \\ +R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This is the reduced row echelon form. ◁

Definition 11.7 (Basic/non-basic column)**.** Let A be a coefficient matrix in reduced row echelon form. We call a column containing a leading entry *basic column* and a column without leading entry *non-basic column*. ◁

Example 11.7. We take the result of the second last example and call the columns of the coefficients $c_1 \dots c_4$:

$$(A \mid b) = \left(\begin{array}{cccc|c} 1 & 0 & 0 & -9 & 1 \\ 0 & 1 & 0 & 5 & 2 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We neglect the constants (right column) and realize that c_1, c_2 and c_3 are basic columns whereas column c_4 is a non-basic column. ◁

11.4. Solutions of an SLE

11.4.1. Three solution behaviours

When trying to find the solution of an SLE we may find

1. no solution,
2. a single solution or
3. an infinite number of solutions.

For two unknowns this may be visualised in a diagram where each equation is represented by a straight line. The solutions are where the lines coincide.

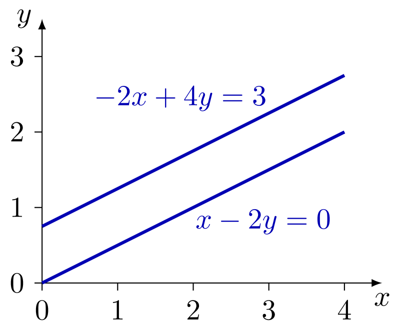
If the lines are parallel but not identical, there is no solution. If the lines intersect, there is a single solution. If the lines coincide, i.e. they have the same slope and position, there exist an infinite number of solutions along the line.

Example 11.8. We look at three examples with two equations and unknowns and visualize the equations as graphs.

- The system of linear equations

$$x - 2y = 0 \qquad -2x + 4y = 3$$

has no solution.

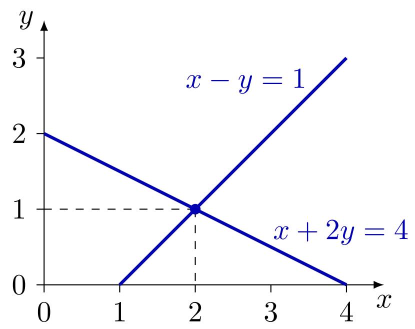


- The system of linear equations

$$x - y = 1 \qquad x + 2y = 4$$

has the single solution

$$x = 2 \qquad y = 1$$

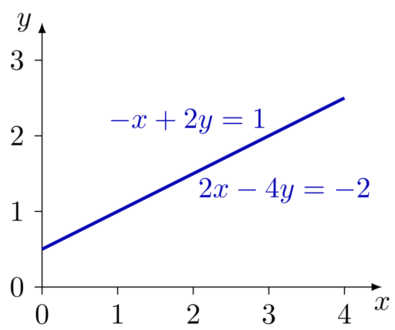


- The system of linear equations

$$-x + 2y = 1 \qquad 2x - 4y = -2$$

has an infinite number of solutions which may be expressed as:

$$y = \frac{x+1}{2}$$



$\triangleleft$

Remark: For an infinite number of solutions one (or more) of the unknowns are used as a parameter to express the other unknowns, see further down.

11.4.2. Solution of a homogeneous SLE

The constant terms of a homogeneous SLE remain zero when performing elementary row operations. Hence, the Gauss-Jordan elimination may be performed on the coefficient matrix only.

A homogeneous SLE has always a trivial solution: All unknowns are zero. Hence, a homogeneous SLE shows only two solution behaviours: a single solution or an infinite number of solutions. Also, if a homogeneous SLE has a single solution, then all unknowns are zero.

If the reduced row echelon form is the identity matrix:

$$\mathrm{rref}(A) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

or an identity matrix with some zero-rows underneath:

$$\mathrm{rref}(A) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

then there exist only the single solution where all unknowns are zero. Here all columns of the matrix are basic columns.

If the reduced row echelon form has less non-zero rows than columns (i.e. unknowns) the SLE has an infinite number of solutions. Here the coefficient matrix contains at least one non-basic column.

11.4.3. Solution of an inhomogeneous SLE

We now investigate the influence of the solution behaviour of an inhomogeneous SLE on the reduced row echelon form of the extended coefficient matrix $\mathrm{rref}(A \mid b)$.

If the reduced row echelon form of the extended coefficient matrix has one or more rows where only the constant is non-zero

$$0 \cdot x_1 + 0 \cdot x_2 + \ldots + 0 \cdot x_n \neq 0$$

we have a contradiction. Hence, the SLE has no solution.

If there is no contradiction and the coefficients of the rref looks like the identity matrix:

$$\text{rref}(A \mid b) = \left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & b_1 \\ 0 & 1 & \cdots & 0 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_n \end{array}\right)$$

or the identity matrix with some zero-rows underneath:

$$\text{rref}(A) = \left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & b_1 \\ 0 & 1 & \cdots & 0 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_n \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{array}\right)$$

then, there exists a single solution. In this situation all columns of the coefficient matrix are basic columns.

If the reduced row echelon form does not contradict itself, and there are less non-zero rows than unknowns, then there are an infinite number of solutions. Here the coefficient matrix contains at least one non-basic column.

Example 11.9. From the previous example we get the reduced row echelon form of the extended coefficient matrices:

- $\left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 3 \end{array}\right)$

  The last row leads to a contradiction and, hence, the SLE has no solution.

- $\left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right)$

  We see the identity matrix for the coefficients, i.e. for the coefficients there are basic columns only. Hence, the SLE has a single solution.

- $\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 0 \end{array}\right)$

  There is no contradiction and row 1 has two non-zero coefficients, i.e. for the coefficients there exist et least one non-basic column. Hence, the SLE has an infinite number of solutions.

$\triangleleft$

Example 11.10. We want to solve the following SLE:

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & = & 1 \\ x_1 & + & x_2 & + & x_3 & = & 2 \\ 2x_1 & + & 3x_2 & + & 4x_3 & = & 3 \end{array}$$

We convert the extended coefficient matrix into its reduced row echelon form:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & 3 & 4 & 3 \end{array}\right) \begin{array}{l} \\ -R_1 \\ -2R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & -1 & -2 & 1 \end{array}\right) \begin{array}{l} +2R_2 \\ *(-1) \\ -R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This is the reduced row echelon form. It has no contradiction (row with zero coefficients only and non-zero constant). The coefficients show two basic columns (one and two) and one non-basic column (three). Hence, the SLE has an infinite number of solutions. $\triangleleft$

Remark: For a solvable SLE every non-basic column adds a degree of freedom to the solution. I.e. we may treat the unknowns of the non-basic columns as parameters and express all unknowns by these parameters.

Example 11.11. We take the extended coefficient matrix of the previous example in reduced row echelon form:

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Columns 1 and 2 are basic and column 3 is non-basic. Therefore we treat the unknown variable x_3 as a parameter, e.g. a, and express all unknowns by this parameter. For the first two rows we get:

$$x_1 - a = 3, \qquad x_2 + 2a = -1$$

and we express the solution by:

$$x_1 = a + 3, \quad x_2 = -2a - 1, \quad x_3 = a$$

I.e. this SLE has one degree of freedom. $\triangleleft$

Example 11.12. We found for an SLE with the unknowns $x_1 \ldots x_5$ the following reduced row echelon form:

$$\left(\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -1 & -3 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Columns 1, 3 and 4 are basic, columns 2 and 5 are non-basic. Treating x_2 and x_5 as parameters a and b, respectively, we get:

$$x_1 = 2a - b + 2 \qquad x_2 = a \qquad x_3 = b - 3$$
$$x_4 = 2b + 1 \qquad x_5 = b$$

I.e. the degree of freedom is 2. ◁

11.5. Problems

Problem 11.1: Solve the following SLE by Gauss-Jordan elimination using the extended coefficient matrix.

1. $\begin{array}{rcrcrcr} 2x & + & y & + & z & = & 7 \\ x & + & 2y & + & z & = & 8 \\ x & + & y & + & 2z & = & 9 \end{array}$

2. $\begin{array}{rcrcrcr} a & - & b & + & c & = & 1 \\ a & + & b & - & c & = & 1 \\ a & + & b & + & c & = & -1 \end{array}$

3. $\begin{array}{rcrcrcr} 5x_1 & + & 4x_2 & + & 3x_3 & = & 4 \\ 3x_1 & + & 2x_2 & + & x_3 & = & 4 \\ x_1 & - & x_2 & - & 2x_3 & = & 6 \end{array}$

4. $\begin{array}{rcrcrcr} 2a & + & 4b & - & c & = & 2 \\ 5a & - & 3b & + & 2d & = & 3 \\ 4a & - & 2c & + & d & = & -1 \\ b & + & 2c & - & d & = & 2 \end{array}$

5. $\begin{array}{rcrcrcrcr} 4x_1 & + & 3x_2 & + & 2x_3 & + & x_4 & = & -2 \\ x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & = & 2 \\ x_1 & + & 2x_2 & + & x_3 & + & 2x_4 & = & 2 \\ x_1 & + & x_2 & + & 2x_3 & + & 3x_4 & = & 1 \end{array}$

6. $\begin{array}{rcrcrcrcr} 2u & + & v & + & w & + & x & = & 6 \\ u & - & v & + & w & - & 3x & = & 5 \\ 2u & + & v & - & w & - & 2x & = & 3 \\ u & + & 2v & - & 2w & - & x & = & 0 \end{array}$

Problem 11.2: Solve the following SLE by changing them into reduced row echelon form. Explain the results.

1. $\begin{array}{rcrcr} x & + & y & = & 1 \\ y & + & z & = & -1 \\ x & - & z & = & 1 \end{array}$

2. $\begin{array}{rcrcr} a & + & b & = & 1 \\ b & + & c & = & 1 \\ a & + & c & = & 0 \end{array}$

3. $\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 4 \\ 3x_1 & + & 2x_2 & + & x_3 & = & 8 \\ x_1 & + & x_2 & + & x_3 & = & 3 \end{array}$

4. $\begin{array}{rcrcrcrcr} a & + & 2b & + & 3c & + & 4d & = & -7 \\ 3a & + & 2b & + & c & + & d & = & 5 \\ 3a & + & 2b & + & c & + & 2d & = & 3 \\ a & + & b & + & c & + & d & = & 0 \end{array}$

5. $\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & - & x_3 & - & 2x_4 & = & -1 \\ 3x_1 & + & x_2 & - & x_3 & - & 3x_4 & = & -2 \\ x_1 & - & x_2 & - & x_3 & + & x_4 & = & 0 \\ x_1 & + & x_2 & - & x_3 & - & x_4 & = & 1 \end{array}$

12. Matrices

12.1. Introduction

In the previous chapter we already expressed a system of linear equations SLE by a matrix and two column-vectors: $A \cdot x = b$. However, we used it only as a technique to write our SLE without extra addition and equality signs $+$ and $=$.

In this chapter we will see the meaning behind this notation an learn basic operations on matrices. We will see that the inverse matrix A^{-1} is a useful tool to evaluate the unknowns for many different constants b. And finally, we will look again on the solution behaviour of SLEs by analysing its (extended) coefficient matrix.

12.2. Definition and special matrices

Definition 12.1 ($m \times n$-matrix)**.** A rectangular arrangement of numbers $a_{ij} \in \mathbb{K}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, $m, n \in \mathbb{N}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is called an $m \times n$*-matrix*. We call a_{ij} the *elements* of A, m the number of rows and n the number of columns. The set of all possible $m \times n$-matrices is denoted by $M(m \times n, \mathbb{K})$. In short we write

$$A = (a_{ij})_{m,n} \qquad \text{or just} \qquad A = (a_{ij})$$

We call $r_i := (a_{i1}, a_{i2}, \ldots, a_{in})$ the i^{th} *row-vector* of A and write

$$A = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{pmatrix}$$

We call

$$c_j := \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} = (a_{1j}, a_{2j}, \ldots, a_{mj})^T$$

the j^{th} *column-vector* of A and write:

$$A = ((c_1, c_2, \ldots, c_n))$$

The double brackets indicate a matrix in contrast to a vector. ◁

Definition 12.2 (Special matrices)**.** Some special matrices:

- All elements of the *zero-matrix* $0_{m,n} \in M(m \times n, \mathbb{K})$ are zero, e.g.

$$0_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad 0_{3,4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- For a *square-matrix* the number of rows and columns are equal, i.e. $M(n \times n, \mathbb{K})$. E.g.

$$A = (a_{ij})_{3,3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

  In short we may write: $A = (a_{ij})_3$

- A *diagonal matrix* is a (usually square) matrix $A = (a_{ij})_{m,n}$ with

$$a_{ij} = \begin{cases} * & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

  where $*$ denotes any value (including zero). E.g.:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

- The *identity matrix* $I_n \in M(n \times n, \mathbb{K})$ or $I_{m,n} \in M(m \times n, \mathbb{K})$ is a diagonal matrix with the diagonal elements being one:

$$a_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

  E.g.:

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad I_{3,2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

We call the column-vectors of the identity matrix *unit-vectors*. I.e. the identity matrix I_n has the unit vectors $e_1, e_2, \ldots, e_n$:

$$I_n = ((e_1, e_2, \ldots, e_n))$$

- A *lower triangular matrix* or *left triangular matrix* denotes a matrix $A = (a_{ij})$ where all elements a_{ij} with $i < j$ are zero, i.e.

$$a_{ij} = \begin{cases} 0 & \text{for } i < j \\ * & \text{for } i \geq j \end{cases}$$

where $*$ denotes any value (including zero). E.g.

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 2 & 5 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \end{pmatrix}$$

- An *upper triangular matrix* or *right triangular matrix* denotes a matrix $A = (a_{ij})$ where all elements a_{ij} with $i > j$ are zero, i.e.

$$a_{ij} = \begin{cases} 0 & \text{for } i > j \\ * & \text{for } i \leq j \end{cases}$$

where $*$ denotes any value (including zero). E.g.

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 4 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 2 \\ 0 & 4 & 1 \end{pmatrix}$$

◁

12.3. Basic matrix operations

Definition 12.3 (Sum of matrices)**.** Two matrices $A = (a_{ij})_{m,n}$ and $B = (b_{ij})_{m,n}$ are added by adding their elements:

$$A + B = (a_{ij} + b_{ij})_{m,n}$$

◁

Example 12.1.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 7 & 6 \\ 3 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 8 \\ 6 & 7 \\ 5 & 7 \end{pmatrix}$$

◁

Definition 12.4 (Multiplication with scalar)**.** A matrix $A = (a_{ij})_{m,n}$ is multiplied by a scalar $\lambda \in \mathbb{K}$ by multiplying its elements with λ:

$$\lambda \cdot A = (\lambda \cdot a_{ij})_{m,n}$$

◁

Example 12.2.

$$3 \cdot \begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 3 \\ 9 & 6 & 3 \end{pmatrix}$$

◁

Definition 12.5 (Transpose of a matrix)**.** The *transpose* of a matrix $A = (a_{ij})_{m,n}$ is a $n \times m$-matrix with switched rows and columns, i.e.

$$A^T = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

◁

Example 12.3. With

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

we get

$$A^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

◁

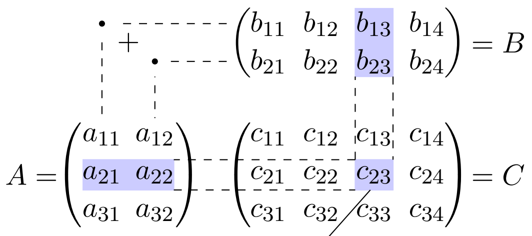
12.4. Matrix multiplication

Definition 12.6 (Matrix product)**.** Let $A = (a_{ij})_{m,n}$ and $B = (b_{ij})_{n,p}$ be two matrices. We define the product $C = (c_{ij})_{m,p}$ of A and B by

$$C = A \cdot B$$
$$(c_{ij})_{m,p} = \left(\sum_{k=1}^{n} a_{ik} b_{kj} \right)_{m,p}$$

◁

Remark: The number of columns of the left matrix must equal the number of rows of the right matrix. The product may be written without the dot: $A \cdot B = AB$.



$$c_{23} = a_{21} \cdot b_{13} + a_{22} \cdot b_{23}$$

Example 12.4. With $A = \begin{pmatrix} 2 & 1 & 6 \\ 4 & 3 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 \\ 4 & 5 \\ 2 & 1 \end{pmatrix}$ what is $A \cdot B$?

With

$$A \cdot B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$
$$= \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

we have:

$$\begin{aligned} c_{11} &= a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = 22 \\ c_{12} &= a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} = 15 \\ c_{21} &= a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} = 34 \\ c_{22} &= a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} = 28 \end{aligned}$$

Hence: $\quad A \cdot B = \begin{pmatrix} 22 & 15 \\ 34 & 28 \end{pmatrix}$ ◁

Theorem 12.7 (Calculation rules for matrix product)**.** For the matrix product we have:

- With $A, B \in M(m \times n, \mathbb{K})$ and $C, D \in M(n \times p, \mathbb{K})$ we have:

  $A(C + D) = AC + AD$ and

  $(A + B)C = AC + BC$

- With $A \in M(m \times n, \mathbb{K})$ and $B \in M(n \times p, \mathbb{K})$ and $\lambda \in \mathbb{K}$ we have:

  $A(\lambda B) = (\lambda A)B = \lambda(AB)$

- With $A \in M(m \times n, \mathbb{K})$, $B \in M(n \times p, \mathbb{K})$ and $C \in M(p \times q, \mathbb{K})$ we have:

  $(AB)C = A(BC)$

- With $A \in M(m \times n, \mathbb{K})$ we have:

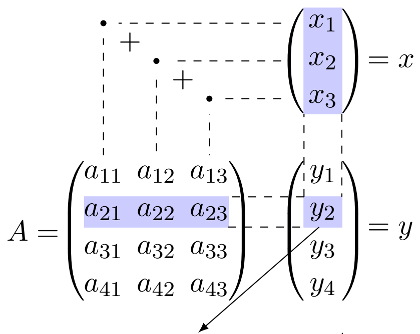
  $A \cdot I_n = A$ and $I_m \cdot A = A$

◁

Remark: I.e. matrix multiplication is distributive over addition, is associative and has the identity matrix as neutral factor. However, the law of commutativity does not hold!

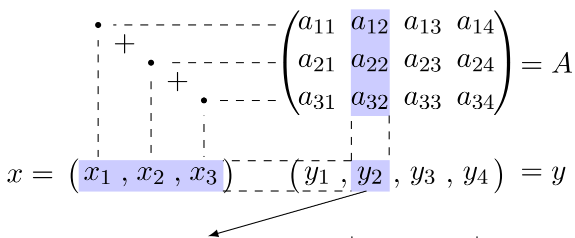
Column- or row-vectors may be looked at as matrices with one column or one row, respectively. Hence, we may multiply matrices with vectors under the same conditions as the matrix product.

The product of a matrix with a column-vector results in another column-vector. The number of columns of the matrix must equal the number of elements of the column-vector to be multiplied. The product column-vector has as many elements as the matrix has rows.



$$y_2 = a_{21} \cdot x_1 + a_{22} \cdot x_2 + a_{23} \cdot x_3$$

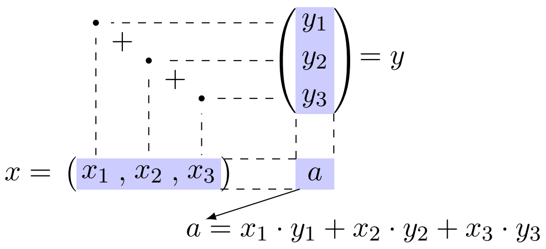
The product of a row-vector with a matrix results in another row-vector. The number of elements of the row-vector to be multiplied must equal the number of rows of the matrix. The product row-vector has as many elements as the matrix has columns.



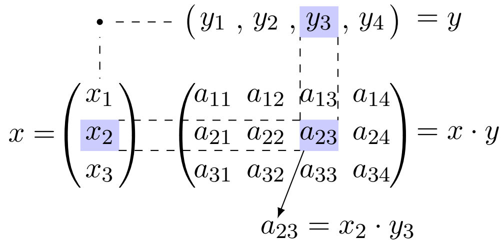
$$y_2 = x_1 \cdot a_{12} + x_2 \cdot a_{22} + x_3 \cdot a_{32}$$

We may also multiply a row-vector with a column-vector and vice versa. Again, the number of columns of the left factor must equal the number of rows of the right factor.

If we multiply a row-vector with a column-vector, both with the same number of elements, the product is a scalar, i.e. an element of $\mathbb{K}$.



Multiplying a column-vector with a row-vector, both with arbitrary number of elements, results in a matrix. The number of rows and columns of the resulting matrix equals the number of elements of column and row vector, respectively.



Definition 12.8 (Inverse matrix)**.**

If for a square matrix $A \in (n \times n, \mathbb{K})$ there exist a matrix $B \in (n \times n, \mathbb{K})$ such that

$$A \cdot B = I_n$$

we say B is the *inverse* of A and denote the *inverse matrix* by A^{-1}. If for a matrix A there exist an inverse matrix A^{-1} we call it an *invertible matrix*. ◁

Theorem 12.9 (Calculation rules for inverse matrices)**.** For matrices $A, B \in M(n \times n, \mathbb{K})$ with existing inverse A^{-1}, B^{-1} we have:

- The inverse of the inverse is the matrix itself:
$$\left(A^{-1}\right)^{-1} = A$$
- The product of a matrix with its inverse is commutative:
$$A \cdot A^{-1} = A^{-1} \cdot A = I_n$$
- The inverse of a product of matrices is the product of the inverse matrices with switched factors:
$$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$$
- The identity matrix is the inverse of itself:
$$I_n{}^{-1} = I_n$$

◁

Remark: We will see further down how to evaluate the inverse of a matrix.

12.5. System of linear equations and matrices

Example 12.5. Let us look at the following equation:

$$\begin{array}{ccccc} A & \cdot & x & = & b \\ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix} & \cdot & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} & = & \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \end{array}$$

We have a matrix $A = (a_{ij})_{4,3}$ for the coefficients, a column-vector $x = (x_1, x_2, x_2)^T$ for the unknowns and another column-vector $b = (b_1, b_2, b_3, b_4)^T$ for the constants. We solve this separately for the elements of b and get

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & = & b_3 \\ a_{41}x_1 & + & a_{42}x_2 & + & a_{43}x_3 & = & b_4 \end{array}$$

which is the set-up of a system on linear equations, SLE, as introduced in the previous chapter. ◁

The example shows that we may use a matrix product to evaluate for a given matrix A and a column-vector x the column-vector b. However, for an SLE we want to do it the other way round: For a given matrix A and a column-vector b we want to derive the column-vector x. We take the matrix equation for the SLE and multiply both sides by the inverse matrix A^{-1}:

$$\begin{aligned} A \cdot x &= b \\ A^{-1} \cdot A \cdot x &= A^{-1} \cdot b \\ I_n \cdot x &= A^{-1} \cdot b \\ x &= A^{-1} \cdot b \end{aligned}$$

Hence, if we find the inverse A^{-1} of the coefficient matrix A of an SLE, we directly may derive the unknowns x for the constants b.

12.6. Matrix inversion

For an invertible matrix $A = (a_{ij})_n$ we want to evaluate the inverse $A^{-1} = (a'_{ij})_n$. By definition we have:

$$A \cdot A^{-1} = I_n$$

We express A^{-1} by its column-vectors $c_j = (a'_{1j}, a'_{2j}, \ldots, a'_{nj})^T$:

$$\begin{aligned} A^{-1} &= \begin{pmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\ a'_{21} & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a'_{n1} & a'_{n2} & \cdots & a'_{nn} \end{pmatrix} \\ &= \left(\left(\begin{pmatrix} a'_{11} \\ a'_{21} \\ \vdots \\ a'_{n1} \end{pmatrix}, \ldots, \begin{pmatrix} a'_{1n} \\ a'_{2n} \\ \vdots \\ a'_{nn} \end{pmatrix}\right)\right) \\ &= ((c_1, c_2, \ldots, c_n)) \end{aligned}$$

We do the same for the identity matrix I_n and express it by its unit-vectors e_j:

$$\begin{aligned} I_n &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \\ &= \left(\left(\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}\right)\right) \\ &= ((e_1, e_2, \ldots, e_n)) \end{aligned}$$

Now we get

$$A \cdot ((c_1, c_2, \ldots, c_n)) = ((e_1, e_2, \ldots, e_n))$$

Column j of a matrix product depends only on the left factor and column j of the right factor. I.e. the column vector e_j of the identity matrix depends only on matrix A and column vector c_j of the inverted matrix A^{-1}. All other columns of the inverted matrix have no influence on column j of the product. Hence, we may separate the equation into a set of n products of a matrix with column-vectors:

$$\begin{array}{ccccc} A & \cdot & c_j & = & e_j \\ A & \cdot & \begin{pmatrix} a'_{1j} \\ a'_{2j} \\ \vdots \\ a'_{nj} \end{pmatrix} & = & \begin{pmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \\ \leftarrow j^{\text{th}} \text{ element} \\ \\ \\ \end{array}$$

for $j = 1, \ldots, n$.

Each of these n products may be solved for the column vector c_j like an SLE. However, since the process is quite similar each time we combine all SLEs into one large extended coefficient matrix:

$$\left(\begin{array}{cccc|cccc} a_{11} & a_{12} & \ldots & a_{1n} & 1 & 0 & \ldots & 0 \\ a_{21} & a_{22} & \ldots & a_{2n} & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} & 0 & 0 & \ldots & 1 \end{array}\right)$$

After performing the Gauss-Jordan elimination we get on the right-hand-side the inverse matrix A^{-1}:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & \ldots & 0 & a'_{11} & a'_{12} & \ldots & a'_{1n} \\ 0 & 1 & \ldots & 0 & a'_{21} & a'_{22} & \ldots & a'_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & a'_{n1} & a'_{n2} & \ldots & a'_{nn} \end{array}\right)$$

Example 12.6. For the SLE

$$\begin{array}{rcrcrcl} 2x & + & y & + & z & = & 7 \\ x & + & 2y & + & z & = & 8 \\ x & + & y & + & 2z & = & 9 \end{array}$$

we want to derive the inverse coefficient matrix. I.e. we neglect the constants in the first place and use them only at the end together with the inverse matrix to evaluate the unknowns.

We create the extended coefficient matrix and apply Gauss-Jordan elimination:

$$\begin{aligned} &\left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \cdot\frac{1}{2} \\ \\ \\ \end{array} \\ \Rightarrow &\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ -R_1 \\ -R_1 \end{array} \\ \Rightarrow &\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right) \begin{array}{l} \\ \cdot\frac{2}{3} \\ \\ \end{array} \\ \Rightarrow &\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right) \begin{array}{l} -\frac{1}{2}R_2 \\ \\ -\frac{1}{2}R_2 \end{array} \\ \Rightarrow &\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{array}\right) \begin{array}{l} \\ \\ \cdot\frac{3}{4} \end{array} \\ \Rightarrow &\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right) \begin{array}{l} -\frac{1}{3}R_3 \\ -\frac{1}{3}R_3 \\ \\ \end{array} \end{aligned}$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right)$$

Thus, we get the inverse matrix:

$$A^{-1} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

With the inverse matrix A^{-1} and the constants $b = (7, 8, 9)^T$ we evaluate the three unknowns:

$$\begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

I.e.

$$\begin{aligned} x &= \tfrac{3}{4} \cdot 7 - \tfrac{1}{4} \cdot 8 - \tfrac{1}{4} \cdot 9 = 1 \\ y &= -\tfrac{1}{4} \cdot 7 + \tfrac{3}{4} \cdot 8 - \tfrac{1}{4} \cdot 9 = 2 \\ z &= -\tfrac{1}{4} \cdot 7 - \tfrac{1}{4} \cdot 8 + \tfrac{3}{4} \cdot 9 = 3 \end{aligned}$$

◁

Example 12.7. The SLE

$$\begin{array}{rcrcrcr} 2x & + & y & + & z & = & 4 \\ x & + & 2y & + & z & = & 3 \\ x & + & y & + & 2z & = & 1 \end{array}$$

is equal to the SLE of the previous example except for the constant vector b. Hence, to derive the unknowns we may take the inverse matrix A^{-1} of the previous example and the constant vector $b = (4, 3, 1)^T$:

$$\begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

i.e. $x = 2, \quad y = 1 \quad \text{and} \quad z = -1.$ ◁

12.7. Rank of a matrix

If a row-vector of a matrix can be expressed by a sum of multiples of other row-vectors, we say the row is *linear dependent* on the other rows. In terms of solving an SLE such a row-vector does not contain any information that is not contained in the other row-vectors. We express this by a general definition of linear dependence:

Definition 12.10 (Linear dependence and independence)**.** If for a set of vectors $v_1, \ldots, v_m \in \mathbb{K}^n$ there exist a set of scalars $\lambda_1, \ldots, \lambda_m \in \mathbb{K}$ with at least one of them not being zero where

$$\lambda_1 \cdot v_1 + \lambda_2 \cdot v_2 + \ldots + \lambda_m \cdot v_m = 0$$

we say these vectors are *linear dependent.* If the above equation holds only for the trivial case, i.e. all scalars are zero, we say the vectors are *linear independent.* ◁

When changing into its reduced row echelon form, a matrix with linear dependent rows will show zero rows whereas a matrix with linear independent rows will show no zero rows.

Hence, the reduced row echelon form of a matrix shows by its zero rows whether rows of the matrix are linear dependent or not.

Linear dependent rows of an SLE may be removed without changing the values of its unknowns.

Theorem 12.11 (Number of linear independent rows and columns)**.** For a matrix $A \in M(m \times n, \mathbb{K})$ the number of linear independent rows and columns are equal. ◁

Definition 12.12 (Rank of a matrix)**.** We call the number of linear independent rows (or columns) the *rank of a matrix.*

For a matrix A we write: $\operatorname{rank}(A)$. ◁

Remark: The rank of a matrix equals the number of non-zero rows in its reduced row echelon form.

Theorem 12.13 (Rank and solutions of an SLE)**.** Let $A \in M(m \times n, \mathbb{K})$ be the coefficient matrix and $(A \mid b) \in M(m \times n+1, \mathbb{K})$ be the extended coefficient matrix of an SLE with m equations and n unknowns. The SLE has

- no solution if
$$\operatorname{rank}(A) \neq \operatorname{rank}(A \mid b)$$
- a single solution if
$$\operatorname{rank}(A) = \operatorname{rank}(A \mid b) = n$$
- an infinite number of solutions if
$$\operatorname{rank}(A) = \operatorname{rank}(A \mid b) < n$$

◁

Example 12.8. We look at the following SLE:

$$\begin{array}{rcrcr} x & + & y & = & 1 \\ y & + & z & = & -1 \\ x & - & z & = & 1 \end{array}$$

and change its extended coefficient matrix into its reduced row echelon form:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & 1 \end{array}\right) \begin{array}{l} \\ \\ -R_1 \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 0 \end{array}\right) \begin{array}{l} -R_2 \\ \\ +R_2 \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{array}\right) \begin{array}{l} \\ \\ \cdot(-1) \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

With A as the coefficient matrix and $(A \mid b)$ as the extended coefficient matrix we get:

$$\operatorname{rank}(A) = 2 \qquad \operatorname{rank}(A \mid b) = 3$$

Since $\operatorname{rank}(A) \neq \operatorname{rank}(A \mid b)$ the SLE has no solution. $\lhd$

Example 12.9. We look at the following SLE:

$$\begin{array}{rcrcrcr} a & + & 2b & + & 3c & = & 0 \\ 3a & + & 2b & + & c & = & 4 \\ a & + & b & + & c & = & 1 \end{array}$$

Changing its extended coefficient matrix into reduced row echelon form we get:

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

With A as the coefficient matrix and $(A \mid b)$ as the extended coefficient matrix of the SLE we get:

$$\operatorname{rank}(A) = 2 \qquad \operatorname{rank}(A \mid b) = 2$$

Since the rank of A and $(A \mid b)$ are equal and they are less than the number of unknowns we get an infinite number of solutions. $\lhd$

Theorem 12.14 (Rank and basic columns)**.** The rank of a matrix A equals its number of basic columns of $\operatorname{rref}(A)$. $\lhd$

12.8. Problems

Problem 12.1: Which of the following statements are true?

1. A diagonal matrix is an identity matrix.
2. The transpose of a square matrix is a square matrix.
3. A diagonal matrix is a triangular matrix.
4. A matrix which is upper and lower triangular is an identity matrix.
5. An identity matrix is a diagonal matrix.
6. A left triangular matrix is an upper triangular matrix.
7. The transpose of an upper triangular matrix is a left triangular matrix.
8. A matrix which is upper and lower triangular is a diagonal matrix.
9. A zero matrix is a triangular matrix.
10. The transpose of an identity matrix is a diagonal matrix.

Problem 12.2: Let $A \in M(2 \times 3, \mathbb{K})$ and $B, C \in M(3 \times 4, \mathbb{K})$. Which of the following statements are true?

1. $A \cdot B \in M(2 \times 4, \mathbb{K})$
2. $B \cdot C \in M(3 \times 4, \mathbb{K})$
3. $A + B \in M(2 \times 4, \mathbb{K})$
4. $B + C \in M(3 \times 4, \mathbb{K})$
5. $A \cdot (B + C) \in M(2 \times 4, \mathbb{K})$
6. $B \cdot C^T \in M(3 \times 3, \mathbb{K})$
7. $B \cdot C^T \in M(4 \times 4, \mathbb{K})$
8. $B^T \cdot C \in M(3 \times 3, \mathbb{K})$
9. $B^T \cdot C \in M(4 \times 4, \mathbb{K})$

Problem 12.3: With

$$x = (1, 2, 3) \qquad y = (3, 3, 1)^T$$

perform the following multiplications:

1. $x \cdot y$
3. $x^T \cdot y^T$

2. $y \cdot x$ 4. $y^T \cdot x^T$

Problem 12.4: With

$$x = (1, 2, 3)$$
$$y = (3, 4, 1, 2)^T$$
$$A = \begin{pmatrix} 2 & 1 & 3 & 2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

perform the following multiplications:

1. A^T
2. $x \cdot A$
3. $A \cdot y$
4. $A \cdot I_4$

Problem 12.5: Evaluate the following products:

1. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix} \cdot \begin{pmatrix} 2 & 4 \\ 5 & 4 \\ 1 & 3 \end{pmatrix}$

3. $\begin{pmatrix} 2 & 4 \\ 5 & 4 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix}$

Problem 12.6: Evaluate the inverse of the following matrices.

1. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

2. $\begin{pmatrix} 1/2 & 3/2 \\ 1/4 & 1/2 \end{pmatrix}$

3. $\begin{pmatrix} 1/2 & 1 & 1/2 \\ 1 & 1/2 & 3/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}$

4. $\begin{pmatrix} 1 & 3 & 2 \\ 2 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix}$

Problem 12.7: Evaluate the inverse of the following matrices.

1. $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 2 & 1 & 0 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 2 & -1 & -2 \\ -2 & 1 & -1 & 2 \\ -1 & -2 & 2 & 1 \\ 2 & -1 & 1 & -1 \end{pmatrix}$

3. $\frac{1}{40} \cdot \begin{pmatrix} -9 & 1 & 1 & 11 \\ 1 & 1 & 11 & -9 \\ 1 & 11 & -9 & 1 \\ 11 & -9 & 1 & 1 \end{pmatrix}$

4. $\begin{pmatrix} 1 & -1 & -1 & -2 \\ -1 & 2 & -1 & 1 \\ 1 & -1 & 2 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$

5. $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

Problem 12.8: Is it possible that the rank of the coefficient matrix of an SLE is larger than the number of unknowns?

Problem 12.9: Is it possible that the rank of the extended coefficient matrix of an SLE is larger than the number of unknowns?

Problem 12.10: Evaluate for the following SLEs the rank of the coefficient matrix and the rank of the extended coefficient matrix. What is the consequence for the solution of the SLEs?

1. $$\begin{aligned} 2x - 2y + 2z &= 0 \\ x + 2y + 2z &= 1 \\ 3y + z &= 1 \end{aligned}$$

2. $$\begin{aligned} 2a - 2b + 2c &= 6 \\ a + 2b + 2c &= 5 \\ 2a + b + 3c &= 4 \end{aligned}$$

3. $$\begin{aligned} 2u - 2v + 2w &= 6 \\ u + 2v + 2w &= 5 \\ 2u + v + w &= 4 \\ u - v - w &= -1 \end{aligned}$$

4. $$\begin{aligned} 2a + 3b + 2c + d &= 4 \\ a + 2b + 3c + 2d &= 4 \\ 3a + 2b + c + d &= 4 \\ 2a + b + c + 3d &= 3 \\ a + b + 3c + 2d &= 3 \end{aligned}$$

5. $$\begin{aligned} 2x_1 + 3x_2 + 2x_3 + x_4 &= 4 \\ x_1 + 2x_2 + x_3 + 2x_4 &= 2 \\ 3x_1 + 2x_2 + 3x_3 + x_4 &= 6 \\ 2x_1 + x_2 + 2x_3 + 3x_4 &= 4 \\ x_1 + x_2 + x_3 - 2x_4 &= 2 \end{aligned}$$

6. $$\begin{array}{rcrcrcrcr} -u & + & 2v & + & w & + & 2x & = & 5 \\ 2u & + & 3v & - & w & - & 2x & = & -1 \\ 2u & - & 3v & + & 2w & + & x & = & -1 \\ 3u & - & v & - & 2w & + & x & = & -8 \\ u & + & v & - & 3w & - & x & = & -6 \end{array}$$

13. Determinants

13.1. Introduction

As you may have learned elsewhere, determinants have something to do with matrices. However, the concept of determinants is older than the concept of matrices!

The word *determinant* comes from *determine* and it is about *to determine* whether a system of linear equations (SLE) is solvable. It was decades later that an SLE was formalized by matrices.

The determinant is also useful to determine the surface of a parallelogram or the volume of a parallelepiped. Furthermore, determinants may be used to actually solve an SLE. These are the topics we want to handle in this chapter.

13.2. Determinant of 2×2 matrix

We look at a system of linear equations (SLE) with two equations and two unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

Under which conditions is this SLE solvable with a single solution? We solve the SLE with respect to x_1 and x_2 and get:

$$\begin{aligned} x_1 &= \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}} \\ x_2 &= \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}} \end{aligned}$$

The denominator of both equations are equal and must be non-zero to derive a unique solution for x_1 and x_2, i.e.

$$a_{11}a_{22} - a_{12}a_{21} \neq 0 \quad \text{(cond. for single sol.)}$$

Hence, to determine whether an SLE is uniquely solvable we evaluate the so called determinant:

Definition 13.1 (Determinant of 2×2 matrix)**.** With $A = (a_{ij}) \in M(2 \times 2, \mathbb{K})$ we call

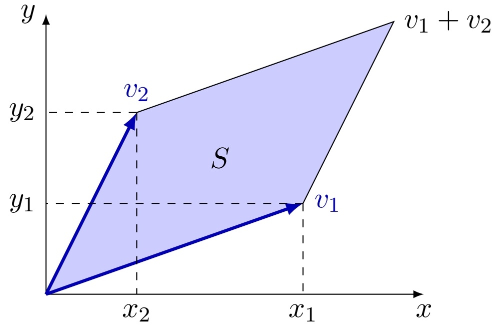
$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

the *determinant* of A. In short we write:

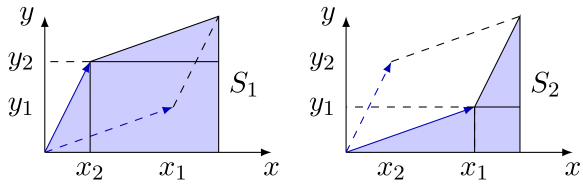
$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

◁

The determinant may be interpreted as the surface S of a parallelogram spanned by the two column-vectors $v_1 = (x_1, y_1)^T$ and $v_2 = (x_2, y_2)^T$.



We separate the parallelogram into an area S_1 below the two upper lines and an area S_2 below the two lower lines.



$$\begin{aligned} S_1 &= \frac{x_2y_2}{2} + x_1y_2 + \frac{x_1y_1}{2} \\ S_2 &= \frac{x_1y_1}{2} + x_2y_1 + \frac{x_2y_2}{2} \\ S &= S_1 - S_2 \\ &= \frac{x_2y_2}{2} + x_1y_2 + \frac{x_1y_1}{2} \\ &\quad - \frac{x_1y_1}{2} - x_2y_1 - \frac{x_2y_2}{2} \\ &= x_1y_2 - x_2y_1 \end{aligned}$$

Using the standard notation of a matrix

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = A$$

we get

$$S = x_1y_2 - x_2y_1 = a_{11}a_{22} - a_{12}a_{21}$$

which looks much like the determinant of A. However, the determinant of A may take negative values. Hence we write:

$$S = |\det(A)|$$

From this graphical representation of the determinant of a 2×2 matrix A we can derive the following properties:

- If the two column-vectors of A are equal, the determinant of A is zero.
- If the two column-vectors of A point exactly in the same or opposite direction, the determinant of A is zero.
- If one of the column-vectors of A is zero, the determinant of A is zero.

13.3. Determinant of 3×3 matrix

We look at an SLE with three equations and three unknowns.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

When solving towards x_1, x_2 and x_3 we get some fairly long equations. However, the three equations all have the same denominator. If this denominator is non-zero the SLE is solvable with a single solution:

$$\begin{aligned} &a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} \neq 0 \\ &\quad\ \text{(condition for single solution)} \end{aligned}$$

Hence, to determine whether the SLE has a single solution we define the determinant:

Definition 13.2 (Determinant of 3×3 matrix)**.** With $A = (a_{ij}) \in M(3 \times 3, \mathbb{K})$ we call

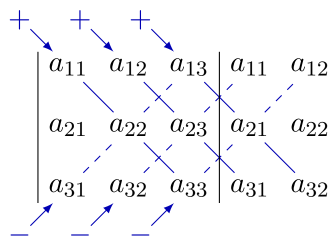
$$\begin{aligned} \det(A) = \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\ = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} \end{aligned}$$

the *determinant* of A. In short we write:

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

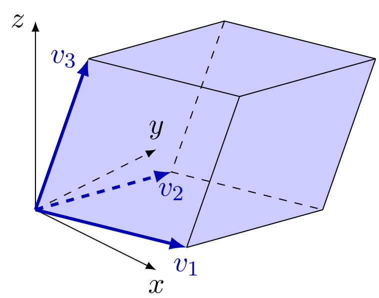
◁

To memorize the equation we write down the 3×3 matrix and repeat column 1 and 2 once more to the right of the matrix. Now we add the products of the three diagonals pointing towards south-east and subtract the products of the three diagonals pointing towards north-east. This is the *rule of Sarrus*.



The absolute determinant of a 3×3 matrix A gives the volume V of a parallelepiped if its spanning vectors v_1, v_2 and v_3 are combined as row- or column-vectors into a matrix, i.e.

$$V = |\det(A)|$$



We observe:

- if all vectors are equal the determinant is zero
- if all vectors are on a line or on a plane the determinant is zero
- if one vector is zero the determinant is zero

Transferred to a matrix we observe: if the columns or rows are linear dependent, the determinant is zero.

13.4. Determinant of n×n matrix

How do we evaluate the determinant for a square matrix with arbitrary size? Unfortunately, the rule of Sarrus can not be applied for matrices larger than 3×3. Hence, we need a general method to evaluate the determinant of a square matrix. To get there we need the term *minor* of a matrix A:

Definition 13.3 (Minor of a matrix)**.** The *minor* $M_{ij} \in \mathbb{K}$ of a matrix $A \in M(n \times n, \mathbb{K})$ is the determinant of matrix A where row i and column j have been removed. ◁

Example 13.1. The matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

has the minors

$$M_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}, \quad M_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}, \quad M_{13} = \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix},$$
$$M_{21} = \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix}, \quad M_{22} = \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix}, \quad M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix},$$
$$M_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}, \quad M_{32} = \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}, \quad M_{33} = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix},$$

i.e.

$$\begin{aligned} M_{11} &= -3, & M_{12} &= -6, & M_{13} &= -3, \\ M_{21} &= -6, & M_{22} &= -12, & M_{23} &= -6, \\ M_{31} &= -3, & M_{32} &= -6, & M_{33} &= -3. \end{aligned}$$

◁

Definition 13.4 (Laplace expansion)**.** Let $A = (a_{ij})_{n,n}$ and M_{ij} its minors. We define the determinant by

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

for any $j \in \{1, \ldots, n\}$ (column expansion) or

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

for any $i \in \{1, \ldots, n\}$ (row expansion). ◁

Example 13.2. We want to evaluate the determinant of the matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

We use Laplace expansion on the first row:

$$\begin{aligned} \det(A) &= +a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} \\ &= +1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ &= 1 \cdot (-3) - 2 \cdot (-6) + 3 \cdot (-3) = 0 \end{aligned}$$

or on the second column:

$$\begin{aligned} \det(A) &= -a_{12}M_{12} + a_{22}M_{22} - a_{32}M_{32} \\ &= -2 \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5 \cdot \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - 8 \cdot \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \\ &= -2 \cdot (-6) + 5 \cdot (-12) - 8 \cdot (-6) = 0 \end{aligned}$$

◁

Remark: When the determinant of a larger matrix is evaluated it is useful to expand over rows or columns with many zeros.

Example 13.3. We want to evaluate the determinant of the matrix:

$$A = \begin{pmatrix} 2 & 4 & 3 & 0 \\ 0 & 5 & 3 & 0 \\ 3 & 0 & 4 & 2 \\ 0 & 3 & 5 & 0 \end{pmatrix}$$

We chose the last column to apply Laplace expansion:

$$\begin{pmatrix} 2 & 4 & 3 & 0 \\ 0 & 5 & 3 & 0 \\ 3 & 0 & 4 & \boxed{2} \\ 0 & 3 & 5 & 0 \end{pmatrix}$$

The 2 in row 3 is the only non zero value in column 4. It is in row three ($i = 3$) column four ($j = 4$), i.e. with $(-1)^{3+4} = -1$ we get:

$$\det(A) = -2 \cdot \begin{vmatrix} 2 & 4 & 3 \\ 0 & 5 & 3 \\ 0 & 3 & 5 \end{vmatrix}$$

Then we choose the first column to apply once more Laplace expansion:

$$\begin{pmatrix} \boxed{2} & 4 & 3 \\ 0 & 5 & 3 \\ 0 & 3 & 5 \end{pmatrix}$$

For the 2 in the first row and column we get a positive sign, hence:

$$\begin{aligned} \det(A) &= -2 \cdot 2 \cdot \begin{vmatrix} 5 & 3 \\ 3 & 5 \end{vmatrix} \\ &= -2 \cdot 2 \cdot (5 \cdot 5 - 3 \cdot 3) = -64 \end{aligned}$$

◁

Although we can not *see* more than three dimensions we generally may look at the absolute of determinants as the volume of parallelepipeds spanned by the row- or column-vectors of the matrices. This gives us a better understanding for some of the following properties.

13.5. Properties of determinants

Theorem 13.5 (Properties of determinants)**.** Let $A \in M(n \times n, \mathbb{K})$ be a square matrix.

- If two rows or two columns of A are equal: $\det(A) = 0$
- If one row or column of matrix A is zero: $\det(A) = 0$
- If the rows or columns of A are linearly dependent: $\det(A) = 0$

◁

Theorem 13.6 (Determinant and invertible matrix)**.** A matrix A is invertible if and only if its determinant is not zero:

$$\det A \neq 0 \quad \Leftrightarrow \quad A \text{ invertible}$$

◁

Theorem 13.7 (Determinants and elementary row operations)**.** Let $A = M(n \times n, \mathbb{K})$ be a square matrix. We then have for

row switching: With A' being matrix A with two rows exchanged we get:

$$\det(A') = -\det(A)$$

row multiplication: With A' being matrix A with one row multiplied by $\lambda \in \mathbb{K}$ we get:

$$\det(A') = \lambda \det(A)$$

row addition: With A' being matrix A with a multiple of one row added to another row we get:

$$\det(A') = \det(A)$$

◁

Theorem 13.8 (Further properties of determinants)**.** Let $A, B = M(n \times n, \mathbb{K})$ be square matrices and $\lambda \in \mathbb{K}$. We then have

- $\det A = \det A^T$
- $\det(\lambda A) = \lambda^n \det A$
- $\det(A \cdot B) = \det A \cdot \det B$
- $\det(A^{-1}) = (\det A)^{-1} \quad \text{for } \det(A) \neq 0$

◁

Theorem 13.9 (Determinant of special matrices)**.**

triangular matrix. The determinant of a triangular matrix is the product of its diagonal elements:

$$\det(A_{\text{triangle}}) = \prod_{k=1}^{n} a_{kk} = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$$

diagonal matrix. The determinant of a diagonal matrix is the product of its diagonal elements:

$$\det(A_{\text{diagonal}}) = \prod_{k=1}^{n} a_{kk} = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$$

identity matrix. The determinant of an identity matrix is one:

$$\det(I_n) = 1$$

◁

Example 13.4. The determinant of

$$A = \frac{1}{2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix} \quad \text{is}$$

$$\det(A) = \frac{1}{2^3} \cdot 1 \cdot 4 \cdot 6 = 3$$

◁

Example 13.5. For the matrices

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{pmatrix}$$

we want to derive the determinant of their product $\det(A \cdot B)$.

We discover linear dependency for the rows of B (row 3 is the sum of the other two), hence:

$$\det(A \cdot B) = \det(A) \cdot \det(B) = \det(A) \cdot 0 = 0$$

◁

Example 13.6. We were able to convert a square matrix A by elementary row operations into an identity matrix. We did not exchange any rows, multiplied rows by the factors $\lambda_1 = 2$,

$\lambda_2 = \frac{1}{2}$ and $\lambda_3 = \frac{1}{3}$ and performed some row additions. Are we able to derive $\det(A)$?

We investigate the effect of row-operations on the determinant and find:

$$\det(I_n) = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \det(A)$$
$$\det(A) = \frac{\det(I_n)}{\lambda_1 \cdot \lambda_2 \cdot \lambda_3} = \frac{1}{2 \cdot \frac{1}{2} \cdot \frac{1}{3}} = 3$$

◁

13.6. Cramer's rule

Determinants are a useful tool to determine whether an SLE with n equations and n unknowns has a unique solution. But can we use determinants to actually solve the SLE?

First we look at an SLE with two equations and two unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$
$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$
$$A \cdot x = b$$

with the solution:

$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}}$$
$$x_2 = \frac{a_{11} b_2 - a_{21} b_1}{a_{11}a_{22} - a_{21}a_{12}}$$

In the denominator we find $\det(A)$. The numerator equals the determinant of a modified matrix A_i:

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$
$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

For A_i we replaced column i by the constant vector b.

The same is possible for larger matrices:

Theorem 13.10 (Cramer's rule)**.** We consider a system of linear equations

$$A \cdot x = b$$

with $A \in M(n \times n, \mathbb{K})$ as the coefficient matrix, $x = (x_1, \ldots, x_n)^T \in \mathbb{K}^n$ as the vector of unknowns and $b \in \mathbb{K}^n$ as the vector of constants. We then get

$$x_i = \frac{\det(A_i)}{\det(A)}$$

where A_i denotes the matrix A where column i is replaced by the vector of constants b. ◁

Example 13.7. We consider the SLE

$$\begin{aligned} 3x_1 + 2x_2 &= 9 \\ 2x_1 + 3x_2 &= 1 \end{aligned}$$

and get

$$\begin{array}{ccccc} A & \cdot & x & = & b \\ \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} & \cdot & \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & = & \begin{pmatrix} 9 \\ 1 \end{pmatrix} \end{array}$$

Applying Cramer's rule we derive:

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{\begin{vmatrix} 9 & 2 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}} = \frac{9 \cdot 3 - 1 \cdot 2}{3 \cdot 3 - 2 \cdot 2} = 5$$
$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{\begin{vmatrix} 3 & 9 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}} = \frac{3 \cdot 1 - 2 \cdot 9}{3 \cdot 3 - 2 \cdot 2} = -3$$

◁

Although Cramer's rule is a simple method to solve an SLE, it is very laborious to perform. For an SLE with n equations and n unknowns we must evaluate $n + 1$ determinants of $n \times n$ matrices.

Hence, in most situations it is more efficient to bring the extended coefficient matrix into its reduced row echelon form and then to derive the unknowns.

13.7. Problems

Problem 13.1: Derive the determinant of the following matrices.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$$
$$C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$E = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \qquad F = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

Problem 13.2: Determine the area of the parallelograms spanned by the following vectors.

1. $v_1 = (1, 2)$ and $v_2 = (-1, 3)$
2. $v_1 = (-3, 6)$ and $v_2 = (2, -4)$
3. $v_1 = (4, 5)$ and $v_2 = (5, 4)$

Problem 13.3: Derive the determinant of the following matrices.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 2 \\ 3 & 6 & 2 \end{pmatrix} \qquad F = \begin{pmatrix} 5 & 2 & 8 \\ 3 & 4 & 1 \\ 7 & 2 & 6 \end{pmatrix}$$

Problem 13.4: Determine the volume of the parallelepiped spanned by the vectors $v_1 = (1, 2, 3)$, $v_2 = (2, 2, -1)$ and $v_3 = (-1, 3, 3)$.

Problem 13.5: Derive the determinant of the following matrices.

$$A = \begin{pmatrix} 0 & 3 & 3 & 4 \\ 2 & 3 & 4 & 2 \\ 4 & 0 & 0 & 2 \\ 1 & 3 & 5 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 & 4 & 4 \\ 0 & 5 & 3 & 0 \\ 1 & 1 & 2 & 0 \\ 5 & 4 & 3 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 5 & 1 & 5 \\ 2 & 1 & 1 & 2 \\ 4 & 1 & 0 & 4 \\ 2 & 3 & 0 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 5 & 1 & 0 & 3 \\ 1 & 0 & 3 & 2 & 0 \\ 2 & 0 & 4 & 1 & 2 \\ 2 & 0 & 1 & 3 & 0 \\ 1 & 5 & 4 & 5 & 2 \end{pmatrix}$$

Problem 13.6: We applied elementary row operations on a matrix A and converted it into an identity matrix. We multiplied rows by the factors $\lambda_1 = 6$, $\lambda_2 = \frac{1}{8}$, $\lambda_3 = \frac{1}{3}$ and $\lambda_4 = 2$. We exchanged row 1 with row 3 and added multiples of rows to other rows. What is the determinant of A?

Problem 13.7: Let $A, B \in M(4 \times 4, \mathbb{K})$ with $\det(A) = 6$ and $\det(B) = \frac{1}{4}$. Solve the following expressions.

1. $\det(A^T)$
2. $\det(B^T)$
3. $\det(A^{-1})$
4. $\det(B^{-1})$
5. $\det(\frac{1}{2}A)$
6. $\det(2B)$
7. $\det(2 \cdot (A^{-1}))$
8. $\det(2 \cdot (B^{-1}))$
9. $\det((2 \cdot A)^{-1})$
10. $\det((2 \cdot B)^{-1})$
11. $\det(A \cdot B)$
12. $\det(B^T \cdot A)$
13. $\det(A \cdot A^{-1})$
14. $\det(A \cdot B^{-1})$

Problem 13.8: Solve the first three SLEs of problem 11.1 by Cramer's rule.

A. Solutions

A.1. Sets and logic

Solution 1.1:

A	B	1.	2.	3.	4.
0	0	1	1	1	1
0	1	1	1	0	0
1	0	1	1	0	0
1	1	0	0	0	0

A	B	C	5.	6.	7.	8.
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	0	0	1	1
1	0	0	0	0	1	1
1	0	1	1	1	1	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

Solution 1.2:

1.

A	$\overline{A}$	$A \vee \overline{A}$
0	1	1
1	0	1

2.

A	B	$\overline{B}$	$A \wedge \overline{B}$	$\overline{A \wedge \overline{B}}$	$A \Rightarrow B$	$(A \Rightarrow B) \Leftrightarrow \overline{A \wedge \overline{B}}$
0	0	1	0	1	1	1
0	1	0	0	1	1	1
1	0	1	1	0	0	1
1	1	0	0	1	1	1

3.

A	B	$\overline{A}$	$\overline{B}$	$A \wedge B$	$\overline{A} \wedge \overline{B}$	$(A \wedge B) \vee (\overline{A} \wedge \overline{B})$	$A \Leftrightarrow B$	$\ldots \Leftrightarrow \ldots$
0	0	1	1	0	1	1	1	1
0	1	1	0	0	0	0	0	1
1	0	0	1	0	0	0	0	1
1	1	0	0	1	0	1	1	1

Solution 1.3:
Distributivity over $\wedge$:

a	b	c	$b \wedge c$	$a \vee b$	$a \vee c$	$a \vee (b \wedge c)$	$(a \vee b) \wedge (a \vee c)$
0	0	0	0	0	0	0	0
0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

Distributivity over $\vee$:

a	b	c	$b \vee c$	$a \wedge b$	$a \wedge c$	$a \wedge (b \vee c)$	$(a \wedge b) \vee (a \wedge b)$
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	0	1	1	1
1	1	0	1	1	0	1	1
1	1	1	1	1	1	1	1

Solution 1.4:

1. $\overline{\overline{A} \wedge \overline{B}}$ 2. $\overline{A \wedge \overline{B}}$ 3. $\overline{A \wedge \overline{B}} \wedge \overline{\overline{A} \wedge B}$

Solution 1.5:
There are different ways to describe the sentences mathematically.

1. With p for problem we get

$$\forall p : p \text{ has a solution}$$

or with p for problem and s for solution we get

$$\forall p \; \exists s : s \text{ solves } p$$

2. With d for driver with licence and c for car we get

$$\neg\forall d\ \exists c : d \text{ owns } c$$

3. With p for passenger and t for ticket we get

$$\exists p\ \neg\exists t : p \text{ owns } t$$

Solution 1.6:

1. "There is no problem with no solution."

$$\neg\exists p : p \text{ has no solution}$$

or

$$\neg\exists p\ \forall s : s \text{ does not solve } p$$

2. "There exists a person with driving license that does not own a car."

$$\exists p\ \forall c : d \text{ does not own } c$$

3. "Not all passengers in public transport have a valid ticket."

$$\neg\forall p\ \exists t : p \text{ owns } t$$

Solution 1.7: All sets are defined correctly. Some sets are empty.

Solution 1.8:

1. $M_1 = \{x \in \mathbb{N} \mid x = 2\,k, k \in \mathbb{N}\}$
2. $M_2 = \{x \in \mathbb{N} \mid x = z^2, z \in \mathbb{N}\}$
3. $M_3 = \{x \in \mathbb{Q} \mid x = \dfrac{1}{k}, k \in \mathbb{N}\}$
4. $M_4 = \{x \in \mathbb{Q} \mid x = \dfrac{2k}{k!}, k \in \mathbb{N}\}$

Solution 1.9:

1. $M_1 = \{3, 6, 9, \ldots\}$
2. $M_2 = \{1, 2, 6, 24, 120, \ldots\}$
3. $M_3 = \{-2, 2\}$
4. $M_4 = \{-1, 0, 2\}$

Solution 1.10:

set A set B	$\{1,2,3\}$ $\{1,2,3,4\}$	$\{a,b,c,d\}$ $\{a,b,d\}$	$\mathbb{N}$ $\mathbb{N}$	$\mathbb{N}$ $\mathbb{Z}$
$A = B$			$\surd$	
$A \neq B$	$\surd$	$\surd$		$\surd$
$A \subseteq B$	$\surd$		$\surd$	$\surd$
$A \subset B$	$\surd$			$\surd$
$A \supseteq B$		$\surd$	$\surd$	
$A \supset B$		$\surd$		

Solution 1.11:

1. {1,2,3,4,5,a,b,c}
2. {1,2,3}
3. {a,b,c}
4. {a,b,c}
5. {4,5,d,e}

Solution 1.12:

$$\{(1,a),(1,b),(1,c),(2,a),(2,b),\\ (2,c),(3,a),(3,b),(3,c)\}$$

Solution 1.13:

$$\{(0,0,0),(0,0,1),(0,1,0),(0,1,1),\\ (1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$$

Solution 1.14:

1. $\{(1,3),(2,5),(3,7),(4,9),\ldots\}$
2. $\{(1,1),(2,4),(3,9),(4,16),\ldots\}$

Solution 1.15:

subset $R \subseteq A \times A$	Cartesian product	binary relation	function
$\{(1,1),(2,2),(3,3)\}$		$\surd$	$\surd$
$\{(1,1),(1,2),(1,3),$ $(2,1),(2,2),(2,3),$ $(3,1),(3,2),(3,3)\}$	$\surd$	$\surd$	
$\{(1,1),(2,1),(2,2),$ $(3,2),(3,3)\}$		$\surd$	
$\{(1,3),(2,2),(3,1)\}$		$\surd$	$\surd$
$\{(1,2),(2,2),(3,2)\}$		$\surd$	$\surd$

Solution 1.16: The *domain* of a function is the set of all "'input"' elements for which the

function is defined. All elements of the domain are mapped to other elements by the function.

The "'output"' of a function are elements of the *codomain*. However, the codomain may contain elements that are not mapped to by the function.

Solution 1.17: We look at a function $f : D \to C$ with domain D and codomain C.

For a given subset of the domain $X \subseteq D$ the *image set* or just *image* is the set of all elements that are mapped to from X by the function f.

For a given subset of the codomain $Y \subseteq C$ the *inverse image set* or just *inverse image* is the set of all elements of D that are mapped to elements of Y.

Solution 1.18:

term	injective	surjective	bijective
$f : \mathbb{N} \to \mathbb{N}, x \mapsto 3x + 1$	$\surd$		
$f : \mathbb{Z} \to \mathbb{N} \cup \{0\}, x \mapsto x^2$			
$f : \mathbb{N} \to \mathbb{Z}, x \mapsto x$	$\surd$		
$f : \mathbb{Z} \to \mathbb{Z}, x \mapsto x$	$\surd$	$\surd$	$\surd$
$f : \mathbb{R} \to \mathbb{R}, x \mapsto x^3$	$\surd$	$\surd$	$\surd$
$f : \mathbb{R} \to \mathbb{R}_{\geq 0}, x \mapsto \exp(x)$	$\surd$		
$f : \mathbb{R}_{>0} \to \mathbb{R}, x \mapsto \ln(x)$	$\surd$	$\surd$	$\surd$

A.2. Natural numbers and integers

Solution 2.1: See lecture notes

1. def. 2.3
2. remark to def. 2.3
3. def. 2.4
4. def. 2.5
5. def. 2.6

Solution 2.2:

1. $3 \cdot 41$
2. $3 \cdot 107$
3. $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
4. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot$
   $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
5. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$

Solution 2.3:

1. 2^5 2. 24 3. 36 4. x^n 5. 0 6. $f(x)$

Solution 2.4:

a	b	$\gcd(a, b)$	$\operatorname{lcm}(a, b)$
6	8	2	24
30	20	10	60
14	15	1	210
39	117	39	117
123	123	123	123

Solution 2.5: No.

Solution 2.6:

dividend	divisor	quotient	remainder
100	16	6	4
62	7	8	6
156	26	6	0
123	124	0	123
124	123	1	1
1 000	3	333	1

Solution 2.7:

1. $$\begin{aligned} 115 &= 1 \cdot 80 + 35 \\ 80 &= 2 \cdot 35 + 10 \\ 35 &= 3 \cdot 10 + 5 \\ 10 &= 2 \cdot \underline{\underline{5}} + 0 \end{aligned}$$
   $\gcd(80, 115) = 5$

2. $$\begin{aligned} 768 &= 1 \cdot 540 + 228 \\ 540 &= 2 \cdot 228 + 84 \\ 228 &= 2 \cdot 84 + 60 \\ 84 &= 1 \cdot 60 + 24 \\ 60 &= 2 \cdot 24 + 12 \\ 24 &= 2 \cdot \underline{\underline{12}} + 0 \end{aligned}$$
   $\gcd(768, 540) = 12$

3. $$\begin{aligned} 5\,471 &= 4 \cdot 1\,193 + 699 \\ 1\,193 &= 1 \cdot 699 + 494 \\ 699 &= 1 \cdot 494 + 205 \\ 494 &= 2 \cdot 205 + 84 \end{aligned}$$

$$
\begin{aligned}
205 &= 2 \cdot 84 + 37 \\
84 &= 2 \cdot 37 + 10 \\
37 &= 3 \cdot 10 + 7 \\
10 &= 1 \cdot 7 + 3 \\
7 &= 2 \cdot 3 + 1 \\
3 &= 3 \cdot \underline{\underline{1}} + 0
\end{aligned}
$$

$\gcd(5\,471, 1\,193) = 1$

Solution 2.8:

1. $110\,0100_2$
2. 144_8
3. 100_{10}
4. 64_{16}
5. 4_{10}
6. 64_{10}
7. 100_{10}
8. 256_{10}

Solution 2.9:

binary	octal	hexadecimal
11 0111	67	37
1010 0101 1100	5134	A5C
1111	17	F
1101 0110 0011	6543	D63
1 0010 0011	443	123
1010 1011 1100	5274	ABC

Solution 2.10:

1. Induction start (A_1):

$$
\begin{aligned}
\sum_{k=1}^{1} k^3 &= \frac{1^2 \cdot (1+1)^2}{4} \\
1^3 &= \frac{4}{4} \\
1 &= 1
\end{aligned}
$$

Induction step ($A_n \Rightarrow A_{n+1}$): If

$$\sum_{k=1}^{n} k^3 = \frac{n^2 \cdot (n+1)^2}{4}$$

then

$$\sum_{k=1}^{n+1} k^3 = \frac{(n+1)^2 \cdot (n+2)^2}{4}$$

Proof:

$$
\begin{aligned}
\sum_{k=1}^{n+1} k^3 &= \sum_{k=1}^{n} k^3 + (n+1)^3 \\
&= \frac{n^2 \cdot (n+1)^2}{4} + (n+1)^3 \\
&= \frac{n^2 \cdot (n+1)^2 + 4(n+1)^3}{4} \\
&= \frac{(n+1)^2 \cdot (n^2 + 4n + 4)}{4} \\
&= \frac{(n+1)^2 \cdot (n+2)^2}{4}
\end{aligned}
$$

2. Induction start (A_1):

$$
\begin{aligned}
&3 \mid 1^3 + 2 \cdot 1 \\
&3 \mid 3
\end{aligned}
$$

Induction step ($A_n \Rightarrow A_{n+1}$): If

$$3 \mid n^3 + 2n$$

then

$$3 \mid (n+1)^3 + 2(n+1)$$

Proof:

$$
\begin{aligned}
&3 \mid (n+1)^3 + 2(n+1) \\
&3 \mid n^3 + 3n^2 + 3n + 1 + 2n + 2 \\
&3 \mid \underbrace{n^3 + 2n}_{A_n} + \underbrace{3n^2 + 3n + 3}_{3|}
\end{aligned}
$$

3. Induction start (A_4):

$$
\begin{aligned}
4! &> 2^4 \\
24 &> 16
\end{aligned}
$$

Induction step ($A_n \Rightarrow A_{n+1}$): If

$$n! > 2^n$$

then

$$(n+1)! > 2^{n+1}$$

Proof:

$$
\begin{aligned}
(n+1)! &= \underbrace{n!}_{>2^n} \cdot (n+1) \\
&> 2^n \cdot \underbrace{(n+1)}_{>2} \\
&> 2^n \cdot 2 = 2^{n+1}
\end{aligned}
$$

4. Induction start (A_1):

$$
\begin{aligned}
\sum_{k=1}^{1} (2k-1) &= 1^2 \\
1 &= 1
\end{aligned}
$$

Induction step ($A_n \Rightarrow A_{n+1}$): If

$$\sum_{k=1}^{n} (2k-1) = n^2$$

then

$$\sum_{k=1}^{n+1}(2k-1) = (n+1)^2$$

Proof:

$$\begin{aligned}
&\sum_{k=1}^{n+1}(2k-1) \\
&\quad = \sum_{k=1}^{n}(2k-1) + (2(n+1)-1) \\
&\quad = n^2 + 2n + 1 \\
&\quad = (n+1)^2
\end{aligned}$$

Solution 2.11: Let $N(n)$ be the number of possible arrangements of n items.

Induction start (A_1): For one item there is obviously only one option to arrange it in a chain.

$$\begin{aligned} N(1) &= 1 \\ n! &= 1 \qquad \Rightarrow \quad \text{ok} \end{aligned}$$

Induction step ($A_n \Rightarrow A_{n+1}$): If

$$N(n) = n!$$

then

$$N(n+1) = (n+1)!$$

Proof: We assume n items have $n!$ different arrangements. If we add another item there are $n+1$ options to place it in the chain. For each position of the last elelment there are $n!$ options to arrange the other n items. Hence we have:

$$\begin{aligned} N(n+1) &= (n+1)\cdot N(n) = (n+1)\cdot n! \\ &= (n+1)! \quad \Rightarrow \quad \text{ok} \end{aligned}$$

A.3. Rational and real numbers

Solution 3.1:

$1.5 \in \mathbb{Q}$ $1/2 \in \mathbb{Q}$ $\pi \notin \mathbb{Q}$ $\sqrt{3} \notin \mathbb{Q}$ $1.234567 \in \mathbb{Q}$

$-5 \in \mathbb{Q}$ $1/3 \in \mathbb{Q}$ $e \notin \mathbb{Q}$ $\sqrt{4} \in \mathbb{Q}$ $0.\overline{142857} \in \mathbb{Q}$

Solution 3.2: a), d), f), g) and h) are true. b), c), e) and i) are false.

Solution 3.3: If a set X has between any two different elements $a \neq b$ an infinite number of other elements we say the set X is *dense*.

Solution 3.4: There exist sums with an infinite number of rational summands where the sum is not rational. We call such sums *irrational*.

The set of real numbers $\mathbb{R}$ cover all elements of the rational numbers $\mathbb{Q}$ and the set of irrational numbers. I.e. The only difference between $\mathbb{Q}$ and $\mathbb{R}$ that $\mathbb{R}$ contains in addition to $\mathbb{Q}$ these irrational numbers.

Solution 3.5:

1. $A_1 = (0, \infty)$
2. $A_2 = [0, \infty)$
3. $A_3 = (0, 1]$
4. $A_4 = (0, \infty)$
5. $A_5 = [-1, 1]$

Solution 3.6:

set	sup	inf
$A_1 = \{1, 2, 3, 4\}$	4	1
$A_2 = \{-2, 0, 2, 4\}$	4	-2
$A_3 = \{-100, 200\}$	200	-100
$A_4 = \{3, 6, 9, \dots\}$	∞	3
$A_5 = \{\dots, -1, 0, 1, 2, \dots\}$	∞	$-\infty$

Solution 3.7:

set	sup	inf	max	min
$A_1 = (0, \infty)$	∞	0	-	-
$A_2 = [0, \infty)$	∞	0	-	0
$A_3 = (0, 1]$	1	0	1	-
$A_4 = (0, \infty)$	∞	0	-	-
$A_5 = [-1, 1]$	1	-1	1	-1

Solution 3.8:

1. $A_1 = \{0, 1, 2, 3, \dots\}$
2. $A_2 = \{-1, 1\}$
3. $A_3 = \{1, 9, 11, 19, 21, 29, \dots\}$

A.4. Complex numbers

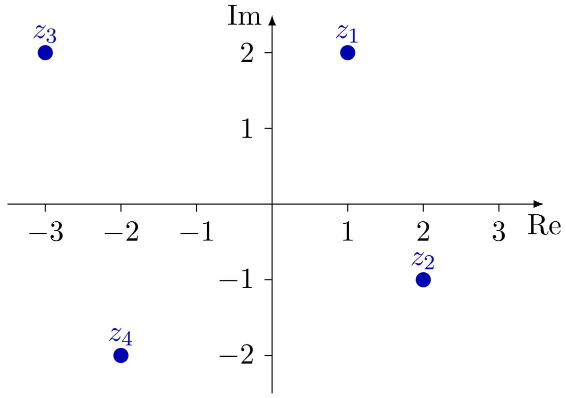
Solution 4.1:

1. -1 2. j 3. 1 4. $-$j 5. $(-1)^n$ 6. j

Solution 4.2:

1. 1 2. 2 3. $-a$ 4. 0

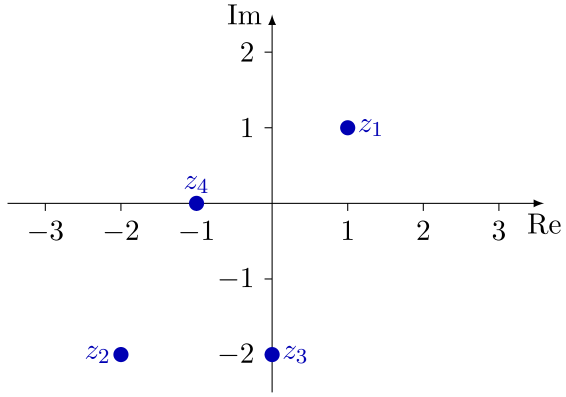
Solution 4.3:



Solution 4.4:

a	b	$\lvert z\rvert$	φ
2	2	$\sqrt{8}$	$\pi/4$
-3	3	$\sqrt{18}$	$3\pi/4$
0	2	2	$\pi/2$
$\sqrt{8}$	$-\sqrt{8}$	4	$-\pi/4$

Solution 4.5:



Solution 4.6:

1. $5 + j$
2. $1 - 5\mathrm{j}$
3. $12 + 5\mathrm{j}$
4. j
5. $-46 + 9\mathrm{j}$
6. -1
7. 13
8. 0

Solution 4.7:

1. $2\,e^{\mathrm{j}7\pi/12}$
2. $4\mathrm{j}$
3. $e^{\mathrm{j}2\pi/3}$
4. $-4\mathrm{j}$
5. $-\frac{1}{2}$
6. 4
7. $\frac{1}{2}\,e^{\mathrm{j}\pi/12}$
8. 2
9. 1
10. 4

Solution 4.8:

1. 4 2. $4\mathrm{j}$ 3. $-1 + \mathrm{j}$ 4. 26 5. 0 6. 13

Solution 4.9:

1. $z_0 = 1,\ z_1 = \mathrm{j},\ z_2 = -1,\ z_3 = -\mathrm{j}$
2. $z_0 = \mathrm{j},\ z_1 = -\mathrm{j}$
3. $z_0 = \sqrt{5},\ z_1 = \sqrt{5}\,\mathrm{j},\ z_2 = -\sqrt{5},\ z_3 = -\sqrt{5}\,\mathrm{j}$
4. $z_0 = 3 + 3\mathrm{j},\ z_1 = -3 + 3\mathrm{j},\ z_2 = -3 - 3\mathrm{j},$ $z_3 = 3 - 3\mathrm{j}$
5. $z_0 = \sqrt{3},\ z_1 = \sqrt{\frac{3}{2}} + \sqrt{\frac{3}{2}}\ \mathrm{j},\ z_2 = \sqrt{3}\ \mathrm{j},$ $z_3 = -\sqrt{\frac{3}{2}} + \sqrt{\frac{3}{2}}\ \mathrm{j}$

   $z_4 = -\sqrt{3},\ z_5 = -\sqrt{\frac{3}{2}} - \sqrt{\frac{3}{2}}\ \mathrm{j},\ z_6 = -\sqrt{3}\ \mathrm{j},$ $z_7 = \sqrt{\frac{3}{2}} - \sqrt{\frac{3}{2}}\ \mathrm{j}$
6. $z_0 = \sqrt{\frac{1}{2}} + \sqrt{\frac{1}{2}}\ \mathrm{j},\ z_1 = -\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{2}}\ \mathrm{j}$

A.5. Sequences

Solution 5.1:

- $(a_n) = (3, 6, 9, 12, 15, \ldots)$
- $(b_n) = (\frac{2}{1}, \frac{3}{3}, \frac{4}{5}, \frac{5}{7}, \frac{6}{9}, \ldots)$
- $(c_n) = (1, 3, 6, 10, 15, \ldots)$
- $(d_n) = (1, 3, 6, 10, 15, \ldots)$
- $(e_n) = (-1, 1, -1, 1, -1, \ldots)$
- $(f_n) = (-\mathrm{j}, -\mathrm{j}, -\mathrm{j}, -\mathrm{j}, -\mathrm{j}, \ldots)$

Solution 5.2: (b_n), (e_n) and (f_n).

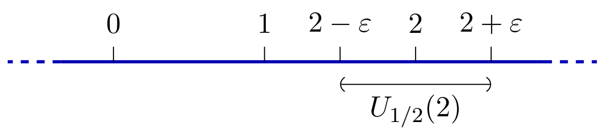
Solution 5.3: Fill the following table with $\surd$ for the appropriate properties of the sequences in problem A.5.

sequence	increasing	strictly increasing	decreasing	strictly decreasing	monotone	strictly monotone
(a_n)	$\surd$	$\surd$			$\surd$	$\surd$
(b_n)			$\surd$	$\surd$	$\surd$	$\surd$
(c_n)	$\surd$	$\surd$			$\surd$	$\surd$
(d_n)	$\surd$	$\surd$			$\surd$	$\surd$
(e_n)						
(f_n)	n.d.	n.d.	n.d.	n.d.	n.d.	n.d.
(g_n)	$\surd$		$\surd$		$\surd$	

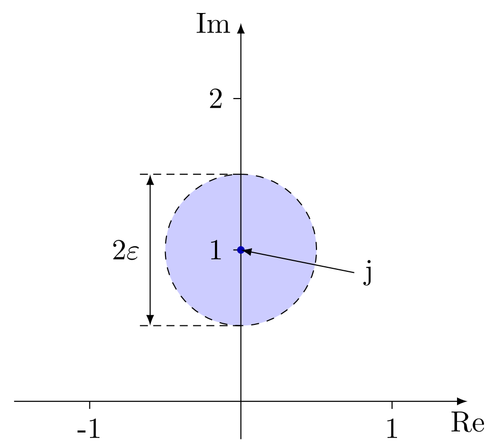
n.d.: Not defined since the function is complex.

Solution 5.4: The given ε-neighbourhood is the set of all real numbers between $\frac{3}{2}$ and $\frac{5}{2}$ excluding the boundaries, i.e.

$$U_\varepsilon(2) = \left(\tfrac{3}{2}, \tfrac{5}{2}\right) \quad \text{for} \quad \varepsilon = \tfrac{1}{2}$$



Solution 5.5: The given ε-neighbourhood is the set of all complex numbers with a distance on the complex plane of less than $\frac{1}{2}$ to the complex number j.



Solution 5.6: If we find for a sequence (a_n) a value a such that for any $\varepsilon \in \mathbb{R}_{>0}$ there is a position n in the sequence from where on all elements have a smaller distance to a than ε then the sequence is *convergent*.

Solution 5.7: (b_n), (f_n) and (g_n) are convergent, all other are divergent.

Solution 5.8:

- $\lim\limits_{n\to\infty} a_n = 1$
- $\lim\limits_{n\to\infty} b_n = \dfrac{1}{2}$
- $\lim\limits_{n\to\infty} c_n = \dfrac{1}{2}$
- $\lim\limits_{n\to\infty} d_n = -\infty$
- $\lim\limits_{n\to\infty} e_n = \infty$

Solution 5.9:

a) $\lim\limits_{n\to\infty} a_n + b_n = \dfrac{3}{2}$

b) $\lim\limits_{n\to\infty} a_n \cdot c_n = \dfrac{1}{2}$

c) $\lim\limits_{n\to\infty} c_n \cdot d_n = -\infty$

d) $\lim\limits_{n\to\infty} \dfrac{e_n}{b_n} = \infty$

A.6. Series

Solution 6.1:

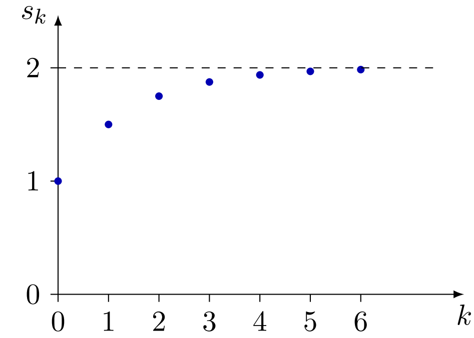
$$a = 15 \qquad b = 2\tfrac{1}{12} \qquad c = 2\tfrac{1}{12} \qquad d = 25$$

Solution 6.2: Evaluate for the following sequences the first 7 elements $(n = 0, 1, \ldots, 6)$ of the sequence of partial sums (s_n):

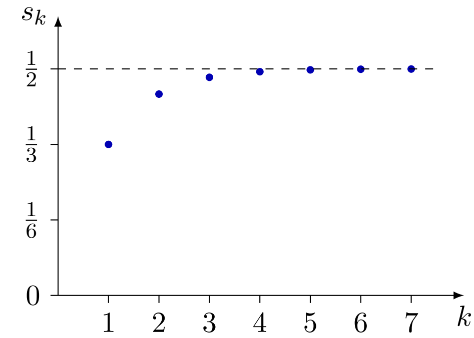
1. $(1, 2, 3, 4, 5, 6, 7, \ldots)$
2. $(0, 1, 4, 9, 16, 25, 36, \ldots)$
3. $(4, 6, 7, 7\frac{1}{2}, 7\frac{3}{4}, 7\frac{7}{8}, 7\frac{15}{16}, \ldots)$
4. $(0, 1, 5, 14, 30, 55, 91, \ldots)$

Solution 6.3:

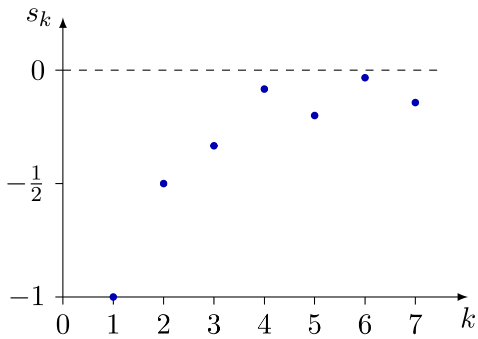
1.



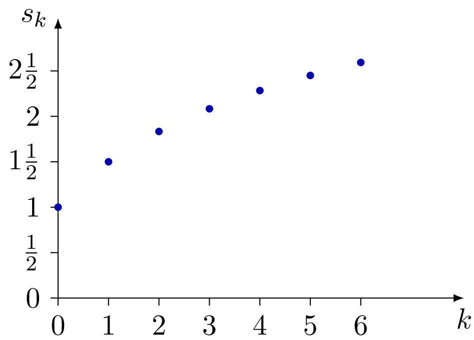
2.



3.



4.



Solution 6.4:

$a = -e \quad b = \frac{1}{3}\ln 2 \quad c = 2e + \ln 2 \quad d = 2\mathrm{j}\ln 2$

Solution 6.5:

$a = 4\,095$	$c = 1\,365$	$e = 2$
$b = 1\frac{2\,047}{2\,048}$	$d = 10$	$f = 1\frac{1}{2}$

Solution 6.6:

1. false	3. true	5. false
2. true	4. true	6. true

Solution 6.7:

1. compare with: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} \quad \Rightarrow \quad$ divergent
2. compare with: $\displaystyle\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k \quad \Rightarrow \quad$ convergent
3. compare with: $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k+1} \quad \Rightarrow \quad$ divergent
4. compare with: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} \quad \Rightarrow \quad$ convergent

Solution 6.8:

1. convergent	3. convergent
2. divergent	4. convergent

Solution 6.9:

1. convergent	3. convergent
2. divergent	4. convergent

A.7. Power series

Solution 7.1:

$f_1(z)$ is a power series
$f_2(z)$ is not a power series
$f_3(z)$ is not a power series
$f_4(z)$ is a power series

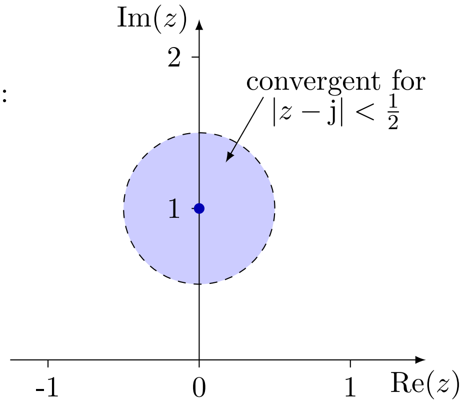
Solution 7.2:

$f_1(z)$ is convergent for $|z| < 1$
$f_2(z)$ is convergent for $|z| < 3$
$f_3(z)$ is convergent for $|z| < 1$

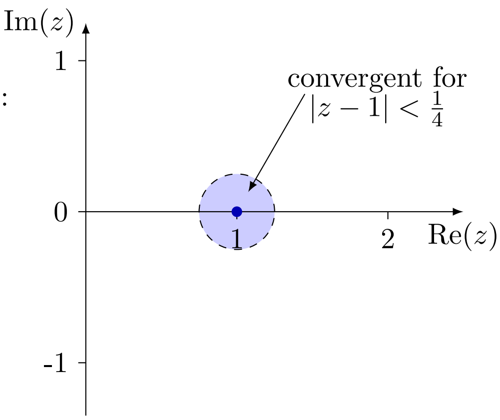
$f_4(z)$ is convergent for $|z - \mathrm{j}| < \frac{1}{2}$
$f_5(z)$ is convergent for $|z - 1| < \frac{1}{4}$

Solution 7.3:

$f_4(z)$:

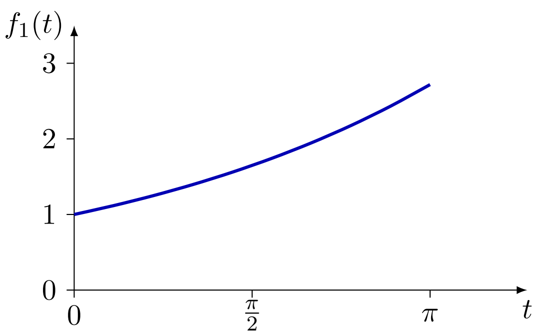


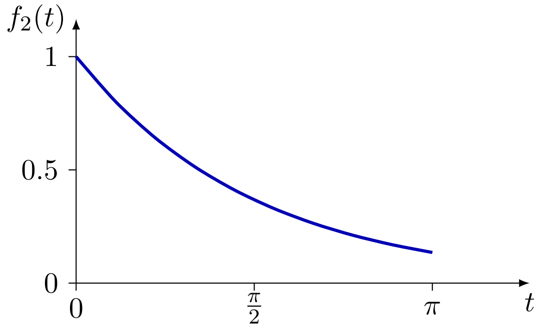
$f_5(z)$:

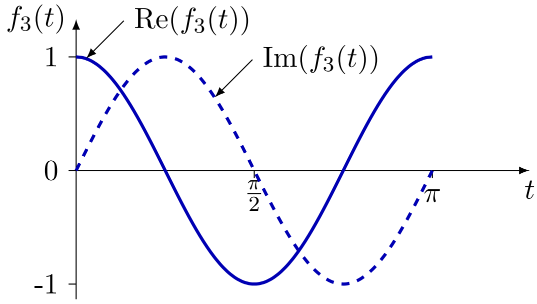


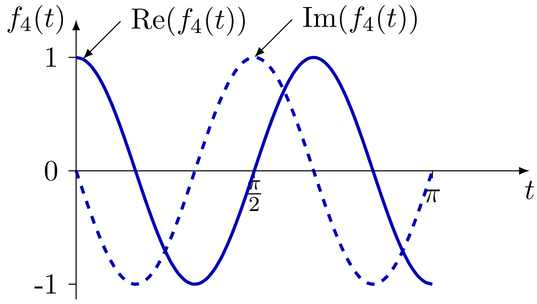
Solution 7.4: The radius of convergence of the exponential function is infinite, i.e. the exponential function is convergent for all $z \in \mathbb{C}$.

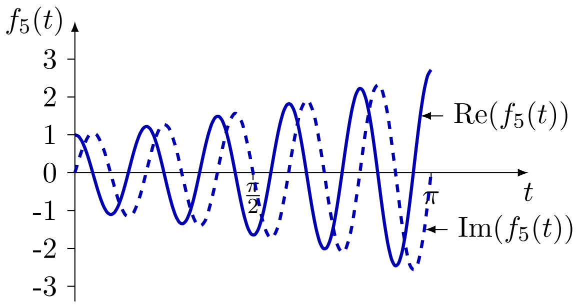
Solution 7.5:

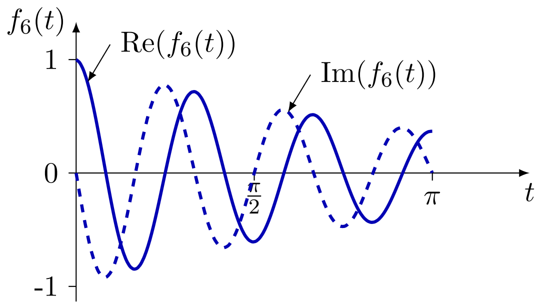












Solution 7.6:

$$
\begin{aligned}
&\sin^2 z + \cos^2 z \\
&\qquad = (\cos z + \mathrm{j} \sin z) \cdot (\cos z - \mathrm{j} \sin z) \\
&\qquad = e^{\mathrm{j}z} \cdot e^{-\mathrm{j}z} = e^{\mathrm{j}z - \mathrm{j}z} = e^0 = 1
\end{aligned}
$$

Solution 7.7:

$$\begin{aligned}
2\mathrm{j}\sin(a-b) &= e^{\mathrm{j}(a-b)} - e^{-\mathrm{j}(a-b)}\\
&= e^{\mathrm{j}a}e^{-\mathrm{j}b} - e^{-\mathrm{j}a}e^{\mathrm{j}b}\\
&= (\cos a + \mathrm{j}\sin a)(\cos b - \mathrm{j}\sin b)\\
&\quad - (\cos a - \mathrm{j}\sin a)(\cos b + \mathrm{j}\sin b)\\
&= \cos a\cos b - \mathrm{j}\cos a\sin b\\
&\quad + \mathrm{j}\sin a\cos b + \sin a\sin b\\
&\quad - \cos a\cos b - \mathrm{j}\cos a\sin b\\
&\quad + \mathrm{j}\sin a\cos b - \sin a\sin b\\
&= 2\mathrm{j}\sin a\cos b + 2\mathrm{j}\cos a\sin b\\
\sin(a-b) &= \sin a\cos b - \cos a\sin b
\end{aligned}$$

$$\begin{aligned}
2\cos(a+b) &= e^{\mathrm{j}(a+b)} + e^{-\mathrm{j}(a+b)}\\
&= e^{\mathrm{j}a}e^{\mathrm{j}b} + e^{-\mathrm{j}a}e^{-\mathrm{j}b}\\
&= (\cos a + \mathrm{j}\sin a)(\cos b + \mathrm{j}\sin b)\\
&\quad + (\cos a - \mathrm{j}\sin a)(\cos b - \mathrm{j}\sin b)\\
&= \cos a\cos b + \mathrm{j}\cos a\sin b\\
&\quad + \mathrm{j}\sin a\cos b - \sin a\sin b\\
&\quad + \cos a\cos b - \mathrm{j}\cos a\sin b\\
&\quad - \mathrm{j}\sin a\cos b - \sin a\sin b\\
&= 2\cos a\cos b - 2\sin a\sin b\\
\cos(a+b) &= \cos a\cos b - \sin a\sin b
\end{aligned}$$

$$\begin{aligned}
2\cos(a-b) &= e^{\mathrm{j}(a-b)} + e^{-\mathrm{j}(a-b)}\\
&= e^{\mathrm{j}a}e^{-\mathrm{j}b} + e^{-\mathrm{j}a}e^{\mathrm{j}b}\\
&= (\cos a + \mathrm{j}\sin a)(\cos b - \mathrm{j}\sin b)\\
&\quad + (\cos a - \mathrm{j}\sin a)(\cos b + \mathrm{j}\sin b)\\
&= \cos a\cos b - \mathrm{j}\cos a\sin b\\
&\quad + \mathrm{j}\sin a\cos b + \sin a\sin b\\
&\quad + \cos a\cos b + \mathrm{j}\cos a\sin b\\
&\quad - \mathrm{j}\sin a\cos b + \sin a\sin b\\
&= 2\cos a\cos b + 2\sin a\sin b\\
\cos(a-b) &= \cos a\cos b + \sin a\sin b
\end{aligned}$$

Solution 7.8:

$$f_1(\varphi) = e^{-\mathrm{j}\varphi} \qquad f_3(\varphi) = e^{\mathrm{j}\varphi} \qquad f_5(\varphi) = \tfrac{1}{2}e^{\mathrm{j}\varphi}$$
$$f_2(\varphi) = \mathrm{j}e^{-\mathrm{j}\varphi} \qquad f_4(\varphi) = \tfrac{1}{2}e^{-\mathrm{j}\varphi} \qquad f_6(\varphi) = e^{0} = 1$$

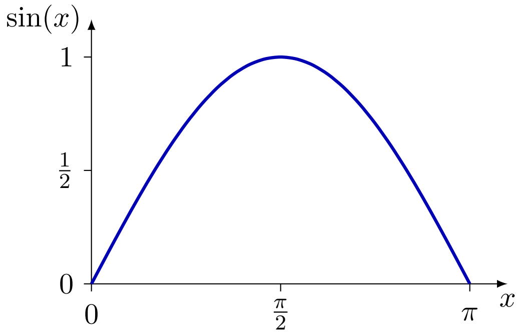
Solution 7.9:

$$x_1 = \tfrac{1}{2} \qquad x_3 = 1 \qquad x_5 = \tfrac{3}{2}$$
$$x_2 = 1 \qquad x_4 = 1 \qquad x_6 = \tfrac{1}{2}$$

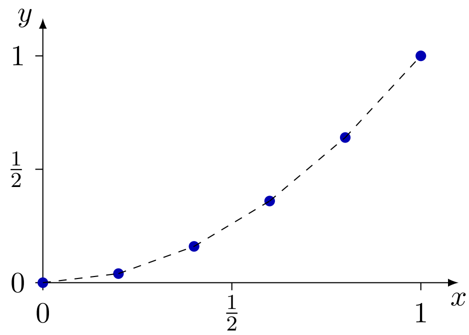
A.8. Functions

Solution 8.1:

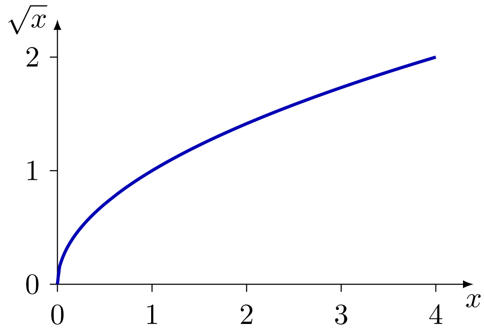
1.



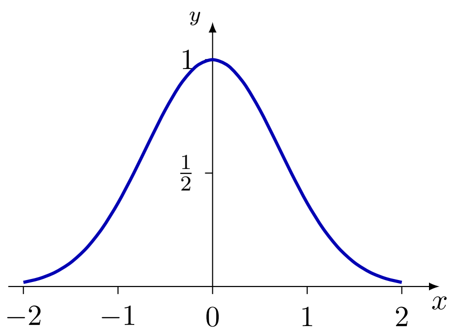
2.



3.



4.



Solution 8.2:

1. strictly decreasing, not bounded, not symmetric, not periodic, no zeros, no extrema.
2. not monotone, bounded, reflection symmetry, not periodic, no zeros, maximum at $x_{\max} = 0$.

3. monotonicity not defined, bounded, not symmetric, periodic over 2π, no zeros, extremum not defined.
4. not monotone, bounded, point symmetry, periodic over 2π, zeros at $x_{\text{zero}} = k\pi$ for $k \in \mathbb{Z}$, maxima at $x_{\max} = 2\pi k + \frac{\pi}{2}$ for $k \in \mathbb{Z}$, minima at $x_{\min} = 2\pi k - \frac{\pi}{2}$ for $k \in \mathbb{Z}$.
5. strictly decreasing, not bounded, not symmetric, not periodic, zero at $x_{\text{zero}} = -1$, no extrema.
6. strictly decreasing, not bounded, not symmetric (in the given interval), not periodic (in the given interval), zero at $x_{\text{zero}} = \frac{\pi}{2}$, no extrema
7. strictly increasing, not bounded, point symmetry, not periodic, zero at $x_{\text{zero}} = 0$, no extrema

Solution 8.3:

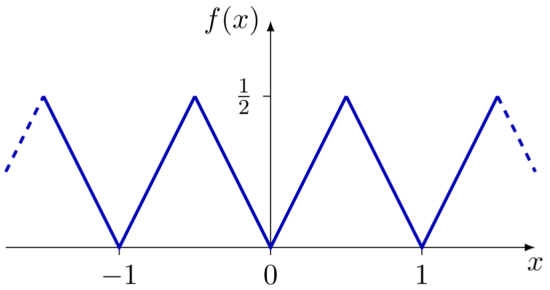
1. $f^{-1} = \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \sqrt[3]{x+1} \end{cases}$
2. $f^{-1} = \begin{cases} \mathbb{R}_{\geq 2} \to \mathbb{R}_{\leq 0} \\ x \mapsto -\sqrt{x-2} \end{cases}$
3. not possible since not bijective.
4. $f^{-1} = \begin{cases} \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2}) \\ x \mapsto \arctan(x) \end{cases}$

Solution 8.4: Only the third function is not continuous.

Solution 8.5: Due to the *intermediate value theorem* the function has at least one zero in the interval $(1, 2)$.

A.9. Differential calculus

Solution 9.1:



The function is continuous, but not differentiable.

Solution 9.2:

1. $-4x$
2. $3e^u$
3. $5x^4 + 3\sqrt{2}x^2$
4. $3x^2 - 2x$
5. $\cos x$
6. $-\sin y$

Solution 9.3:

1. $x^2 + x + 1$
2. $\dfrac{3}{2}x^8 + \dfrac{2}{3}x^2 + \dfrac{4}{5}x^3$
3. $2x + 2y$
4. $2y - 2x$
5. 0
6. $y^2 z^3$
7. $2xyz^3$
8. $3xy^2z^2$

Solution 9.4:

1. $-2x^{-3} - 3x^{-4} = -\dfrac{2}{x^3} - \dfrac{3}{x^4}$
2. $-\dfrac{1}{x^2} - \dfrac{2}{x^3} - \dfrac{3}{x^4}$
3. $2x^{-\frac{1}{3}} + 2x^{-\frac{4}{3}} = \dfrac{2}{\sqrt[3]{x}} + \dfrac{2}{\sqrt[3]{x^4}}$
4. $x + x^{-4} = x + \dfrac{1}{x^4}$
5. $\dfrac{1}{2}x^{-\frac{1}{2}} + \dfrac{1}{2}x^{-\frac{3}{2}} = \dfrac{1}{2\sqrt{x}} + \dfrac{1}{2\sqrt{x^3}}$
6. $2x + 2x^{-3} = 2x + \dfrac{2}{x^3}$

Solution 9.5:

1. e^{2jx}
2. $\dfrac{1}{\cos x} - \tan x \sin x$
3. $\dfrac{\sin x}{\sqrt{x}} + 2\sqrt{x}\cos x$
4. $\dfrac{x^4 + 3x^2 - 4x}{x^4 + 2x^2 + 1}$
5. $\dfrac{2x - x^2}{e^x}$
6. $\dfrac{\cos(x) - \sin(x)\ln(2)}{2^x}$

Solution 9.6:

1. $2xe^{x^2}$
2. $2\pi f \hat{u} \cos(2\pi f t)$
3. $2\pi f j \hat{i} e^{j(2\pi f t + \varphi)}$
4. $2\sin(x)\cos(x)$
5. $\dfrac{1}{x\sqrt{\ln|x^2|}}$
6. $\dfrac{\ln(3)3^x \cos(3^x)}{2\sqrt{\sin(3^x)}}$

Solution 9.7:

1. $12x^2$
2. $\ln^3(3)\,3^x$
3. $\ln^n(\pi)\,\pi^x$
4. $-4\pi^2 f^2 \hat{u}\sin(2\pi f t)$
5. $16\sin(2x)$
6. $(2\pi \mathrm{j} f)^n \hat{u} e^{\mathrm{j}(2\pi f t+\varphi)}$

A.10. Polynomials

Solution 10.1:

1. polynomial of order 4
2. polynomial of order 3
3. no polynomial
4. polynomial of order 2
5. polynomial of order 0 for $x \neq 0$, no polynomial otherwise
6. polynomial of order 100

Solution 10.2:

1. polynomial of order 4
2. polynomial of order 7
3. polynomial of order 4
4. polynomial of order 3
5. polynomial of order 12
6. polynomial of order 24

Solution 10.3:

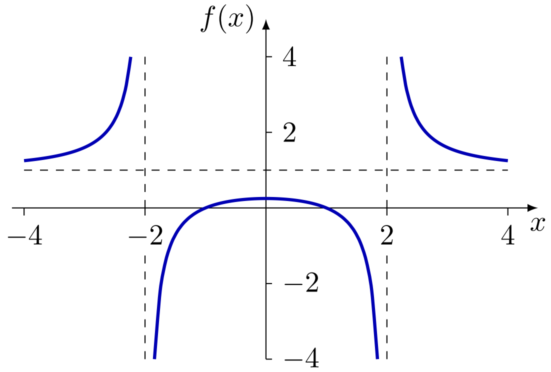
1. x^2-3x+2
2. x^2-4x+3
3. x^2-5x+6
4. x^2+x-2

Solution 10.4:

1. $p_1(x)=(x-1)(x+1)$
2. $p_2(x)=2x(x+1)(x-2)$
3. $p_3(x)=(x-a)(x-b)$
4. $p_4(z)=(z-x)^2$
5. $p_5(x)=(x+1+\mathrm{j})(x+1-\mathrm{j})$
6. $p_6(x)=(x-1+2\mathrm{j})(x-1-2\mathrm{j})(x-1)$

Solution 10.5: The polynomial is at least of second order and has a second zero at $\overline{z_1}$.

Solution 10.6:



Solution 10.7:

1. single zeros at $\{-1,1\}$, single poles at $\{-2,2\}$
2. single poles at $\{-2,1\}$
3. single zeros at $\{-1,1\}$, double pole at $\{2\}$
4. single zeros at $\{-1\}$, single poles at $\{-2\}$, removable discontinuity at $\{2\}$ with limit $\frac{3}{4}$

Solution 10.8:

$$f(x)=\frac{(x-1)(x+1)}{(x-3)(x+3)}=\frac{x^2-1}{x^2-9}$$

Solution 10.9:

1. $f(x)=x^2+1+\dfrac{1}{x^2-1}$
2. $f(x)=x^2-x+1+\dfrac{3-3x}{x^2+x-2}$
3. $f(x)=x^3+x^2+\dfrac{x^2}{x^3-x^2+x-1}$
4. $f(x)=x^3$

Solution 10.10:

1. $f(x)=\dfrac{\frac{1}{2}}{x-1}-\dfrac{\frac{1}{2}}{x+1}$
2. $f(x)=\dfrac{2}{x+2}-\dfrac{1}{(x+2)^2}$
3. $f(x)=\dfrac{1}{2x}+\dfrac{\frac{1}{2}x-1}{x^2+2x+2}$
4. $f(x)=\dfrac{1}{x+1}+\dfrac{\frac{1}{5}}{x-1}+\dfrac{-\frac{1}{5}x-\frac{8}{5}}{x^2+2x+2}$

Solution 10.11:

$$T_2(x) = 1 + x + \frac{x^2}{2}$$

Solution 10.12:

$$T_1(x) = x - 1$$

A.11. System of Linear equations

Solution 11.1:

1. $x = 1, \quad y = 2, \quad z = 3$
2. $a = 1, \quad b = -1, \quad c = -1$
3. $x_1 = 2, \quad x_2 = 0, \quad x_3 = -2$
4. $a = 0, \quad b = 1, \quad c = 2, \quad d = 3$
5. $x_1 = -1, \quad x_2 = 1, \quad x_3 = -1, \quad x_4 = 1$
6. $u = 1, \quad v = 2, \quad w = 3, \quad x = -1$

Solution 11.2:

1. No solution since the last row of the rref shows a contradiction
2. Single solution: $a = 0, \quad b = 1, \quad c = 0$
3. Since the rref shows no contradiction and the number of non-zero rows are less than the number of unknowns we have an infinite number of solutions:

   $x_1 = 2 + x_3, \quad x_2 = 1 - 2x_3$
4. Since the rref shows no contradiction and the number of non-zero rows are less than the number of unknowns we have an infinite number of solutions:

   $a = 3 + c, \quad b = -1 - 2c, \quad d = -2$
5. No solution since the last row of the rref shows a contradiction

A.12. Matrices

Solution 12.1: 2., 3., 5., 7., 8., 9., 10.

Solution 12.2: 1., 4., 5., 6., 9.

Solution 12.3:

1. 12
2. $\begin{pmatrix} 3 & 6 & 9 \\ 3 & 6 & 9 \\ 1 & 2 & 3 \end{pmatrix}$
3. $\begin{pmatrix} 3 & 3 & 1 \\ 6 & 6 & 2 \\ 9 & 9 & 3 \end{pmatrix}$
4. 12

Solution 12.4:

1. $\begin{pmatrix} 2 & 4 & 1 \\ 1 & 3 & 2 \\ 3 & 2 & 3 \\ 2 & 1 & 4 \end{pmatrix}$
2. $(13, 13, 16, 16)$
3. $\begin{pmatrix} 17 \\ 28 \\ 22 \end{pmatrix}$
4. $\begin{pmatrix} 2 & 1 & 3 & 2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} = A$

Solution 12.5:

1. $\begin{pmatrix} 12 & 11 \\ 26 & 25 \end{pmatrix}$
2. $\begin{pmatrix} 17 & 27 \\ 32 & 46 \end{pmatrix}$
3. $\begin{pmatrix} 14 & 20 & 34 \\ 17 & 26 & 49 \\ 10 & 14 & 23 \end{pmatrix}$

Solution 12.6:

1. $\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$
2. $\begin{pmatrix} -4 & 12 \\ 2 & -4 \end{pmatrix}$
3. $\begin{pmatrix} 1 & 3 & -5 \\ 1 & -1 & 1 \\ -1 & -1 & 3 \end{pmatrix}$
4. $\begin{pmatrix} -\frac{3}{8} & \frac{1}{2} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{2} & -\frac{3}{8} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix}$

Solution 12.7:

1. not invertible
2. $\begin{pmatrix} 0 & 3/5 & -1/5 & 1 \\ 1 & 6/5 & 3/5 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}$

4. $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2/3 & 1 & 2/3 & 1/3 \\ 0 & 0 & 1/3 & 1/3 \\ -1/3 & 0 & 0 & -1/3 \end{pmatrix}$

5. $\begin{pmatrix} 1 & 1 & 1 & -2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$

Solution 12.8: No

Solution 12.9: Yes

Solution 12.10:

problem	unknowns	rank(A)	rank($A \mid b$)	solution
1.	3	2	2	infinite no. of sol.
2.	3	2	3	no solution
3.	3	3	3	single solution
4.	4	4	5	no solution
5.	4	3	3	infinite no. of sol.
6.	4	4	4	single solution

A.13. Determinants

Solution 13.1:

$$|A| = -2 \qquad |C| = 0 \qquad |E| = 3$$
$$|B| = -2 \qquad |D| = 1 \qquad |F| = 0$$

Solution 13.2: The volumes of the parallelograms are

1. 5 2. 0 3. 9

Solution 13.3:

$$|A| = -2 \qquad |C| = 18 \qquad |E| = 0$$
$$|B| = 0 \qquad |D| = 1 \qquad |F| = -88$$

Solution 13.4: The volume of the parallelepiped is 23.

Solution 13.5:

$$|A| = 30 \quad |B| = -135 \quad |C| = -40 \quad |D| = 295$$

Solution 13.6: $\det(A) = -2$

Solution 13.7:

1.	6	2.	$\frac{1}{4}$
3.	$\frac{1}{6}$	4.	4
5.	$\frac{3}{8}$	6.	4
7.	$\frac{8}{3}$	8.	64
9.	$\frac{1}{96}$	10.	$\frac{1}{4}$
11.	$\frac{3}{2}$	12.	$\frac{3}{2}$
13.	1	14.	24

Solution 13.8:

1. $x = 1, \quad y = 2, \quad z = 3$
2. $a = 1, \quad b = -1, \quad c = -1$
3. $x_1 = 2, \quad x_2 = 0, \quad x_3 = -2$

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